COMPRENSIÓN DE LAS FRACCIONES COMO MEDIDA POR ESTUDIANTES DE 6º GRADO DE EDUCACIÓN PRIMARIA[1]

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Resumen

La introducción y enseñanza de las fracciones en la escuela primaria privilegia el significado parte-todo. Esta perspectiva tiende a desviar a los estudiantes, ya que pueden entender que el conjunto de los números racionales es una extensión del conjunto de los números naturales. Sin embargo, la investigación ha demostrado que la enseñanza de las fracciones desde una perspectiva de medición, utilizando el material de Cuisenaire, favorece el desarrollo del sentido numérico de las fracciones y permite que conceptos como las fracciones impropias y equivalentes se comprendan fácilmente y se construyan de manera efectiva. Este trabajo investiga actividades matemáticas desencadenadas por tareas exploratorias utilizando el material Cuisenaire para abordar las fracciones en la perspectiva de la medida en el 6º año de la Enseñanza Fundamental, considerando las cuatro fases del Modelo Instruccional 4A en la comprensión de las fracciones. Los resultados revelan que los estudiantes, al manipular, observar y comparar las barras, pasaron de la fase de Acciones Concretas a la Fase de Acciones Formales, dándose cuenta de las relaciones entre las barras y, de esta forma, construyeron ideas matemáticas. Los estudiantes entendieron la diferencia en la magnitud numérica de los números naturales a los números fraccionarios, reconociendo fracciones equivalentes y realizando operaciones, incluso reconociendo que cometieron errores al operar con fracciones porque usaron las propiedades de los números naturales. Además, se les introdujo en el lenguaje algebraico sin generar ninguna carga cognitiva.
***Palabras clave*:** Educación Matemática. Modelo Instruccional 4A. Regletas de Cuisenaire.

A COMPREENSÃO DE FRAÇÕES COMO MEDIDA POR ALUNOS DO 6º ANO DO ENSINO FUNDAMENTAL

Resumo

A introdução e o ensino de frações na escola básica privilegiam o significado parte-todo. Esta perspectiva tende a conduzir os estudantes ao erro, podendo eles compreenderem que o

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conjunto dos números racionais é uma extensão do conjunto dos números naturais, ocasionando confusão com os procedimentos aritméticos de fração, prejudicando a aprendizagem de Álgebra e demais conteúdos matemáticos. No entanto, pesquisas têm evidenciado que o ensino de frações na perspectiva de medição, utilizando o material Cuisenaire, favorece o desenvolvimento do senso numérico de frações e permite que conceitos como frações impróprias e equivalentes sejam facilmente compreendidos e efetivamente construídos. Este trabalho investiga atividades matemáticas desencadeadas por tarefas de natureza exploratória utilizando o material Cuisenaire para abordar frações na perspectiva de medição no 6º ano do Ensino Fundamental, considerando as quatro fases do Modelo 4A-Instrucional na compreensão de frações. Os resultados revelam que os alunos, ao manipular, observar e comparar as barras, passaram da fase de Ações Concretas para a Fase de Ações Formais, percebendo as relações entre as barras e, deste modo, construíram ideias matemáticas. Os alunos compreenderam a diferença da magnitude numérica dos números naturais para os fracionários, reconhecendo as frações equivalentes e realizando operações, inclusive reconhecendo que cometiam erros ao operar com frações por utilizarem propriedades dos números naturais. Além disso, foram introduzidos à linguagem algébrica sem gerar qualquer carga cognitiva.
Palavras-chave: Educação Matemática. Modelo Instrucional 4A. Barras Cuisenaire

UNDERSTANDING FRACTIONS AS MEASUREMENT BY 6TH GRADE STUDENTS FROM ELEMENTARY SCHOOL

Abstract

The introduction and teaching of fractions in Elementary school privileges the part-whole meaning. This perspective tends to lead students to mistakes, as they may understand that the rational numbersset is an extension of the natural numbers set. However, research has shown that teaching fractions from a measurement perspective, using the Cuisenaire material, favors the development of the numerical sense of fractions and allows concepts such as improper and equivalent fractions to be easily understood and effectively constructed. This work investigates mathematical activities resulting from exploratory tasks using the Cuisenaire material to approach fractions in the measurement perspective with 6th-grade Elementary School students, considering the four phases of the 4A-Instructional Model in the understanding of fractions. The results reveal that the students, when manipulating, observing and comparing the bars, moved from the Concrete Actions phase to the Formal Actions Phase, realizing the relations among the bars and, thus built mathematical ideas. Students understood the difference in numerical magnitude from natural numbers to fractional numbers, recognizing equivalent fractions and performing operations, including recognizing their mistakes while operating with fractions as a consequence of using properties of natural numbers. In addition, they were introduced to algebraic language without generating any cognitive load.
Keywords: Mathematics education. 4A Instructional Model. Cuisenaire´s Bars**.**

Introduction

The predominant teaching of fractions emphasizes the part-whole meaning, a perspective that has not been shown to be efficient (POWELL, 2018a), because this approach tends to mislead students in understanding that the set of rational numbers is an extension of the set of rational numbers, natural numbers, causing conceptual and procedural difficulties regarding fractional numbers. This approach makes students to think the same rules for whole numbers apply to fractions. For Siegler *et al.* (2012), another difficulty regarding starting the teaching of fractions in the part-whole perspective is the confusion with the fraction arithmetic procedures, for example, in the problems of addition and subtraction of fractions with the same denominator, when this is kept in the answer, which is different from multiplication and division of fractions. This difficulty with fractions impairs the learning of Algebra and later mathematical content (BOOTH; NEWTON, 2012; SIEGLER *et al.*, 2012; TORBEYNS *et al.*, 2015).

On the other hand, research (POWELL, 2018b; 2019b) with students in the early years, with no previous formal instruction in fractions, has shown that teaching fractions from the measurement perspective, using the Cuisenaire material, favors the development of number sense fractions, and allows concepts, such as improper and equivalent fractions, to be easily understood and effectively constructed. To this end, the researcher and his team developed an instructional model called 4A-Instructional model, which consists of four phases of implementation of a pedagogical approach, the subordination of mathematics teaching to student learning, using Cuisenaire rods. These surveys, however, did not involve students who had already studied fractions. From this, the present work seeks to investigate mathematical activities resulted from exploratory tasks using the Cuisenaire material to approach fractions in the measurement perspective in 6th-grade Elementary students, considering the four phases of the 4A-Instructional Model in the understanding of fractions. So we start our discussions in the next section by discussing understanding numbers.

Understanding numbers and number sense

Understanding numbers involves the ability to approximate and make decisions about their size, which includes recognizing unreasonable approximations and a conceptual understanding of numerical operations, as well as the ability to translate different representations of numbers and choose the most appropriate representation for a given number context (POWELL; ALI, 2018; YANG *et al.*, 2004). According to Mcintosh et al. (1992), number sense refers to the general understanding of numbers and operations, as well as the development of different strategies to deal with them, regardless of people's occupation, resulting in an expectation that numbers are essential and that mathematics has a certain regularity.

For McIntosh *et al.* (1992, p. 5), “a person with good number sense is thinking and reflecting on the numbers, operations and results that are being produced”. The authors debate these notions of number sense within three broad categories: (1) knowledge and facility with numbers, (2) knowledge and facility with operations, and (3) application of knowledge and facility with numbers and operations to computational configurations

Within the *knowledge and facility* category with numbers is the sense of the order of numbers, multiple representations for numbers, system of reference points and sense of relative and absolute magnitude for numbers. The second category, *knowledge and facility with operations*, includes understanding the effects of operations, understanding mathematical properties, and understanding the relationship between operations. Understanding the relationship between the context of the problem and the necessary calculation, awareness that there are multiple strategies, inclination to use an efficient representation and/or method, and inclination to review data and results cautiously are part of the third category, namely, *application of knowledge* and easeness with numbers and operations to computational configurations (MCINTOSH *et al.*, 1992).

Yang, Hsu and Huang (2004) reviewed relevant reports and studies on number sense and defined the components of number sense as follows: 1) understand the meaning of numbers, 2) recognize the magnitude of numbers, 3) use benchmarks appropriately, 4) know the relative effect of the transaction on numbers, and 5) develop estimation strategies and judge the reasonableness of results. For the authors, the development of number sense is of international interest and concern, because a person who has a well-developed number sense has the ability to estimate quantities, make quantitative comparisons, recognize errors in magnitude or measurement judgments (CORSO; DORNELES, 2010). For example, the problem $\frac{6}{13}+\frac{3}{7}$ can be solved in the conventional way, through memorized procedures, often leading students to make mistakes or recognizing that each fraction is slightly less than $\frac{1}{2}$, estimating that the result of the problem should be slightly less than one, and thus recognizing whether the result is reasonable or not.

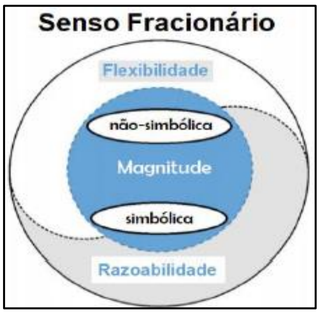
Magnitude is of central importance to numerical understanding. “It describes the size of a quantity - something that can be increased or decreased” (SOUZA; POWELL, 2021, p. 85) and is represented by a number. Powell (2019a, p. 3), supported by Carraher (1996), states that “absolute magnitude or magnitude is the size or extent of an object without considering a comparison or measurement and relative magnitude is the size of a subject object in comparison with another object or measurement with a unit of measurement”. Powell and Ali (2018, p. 235), based on studies by Rodrigues, Dyson, Hansen and Jordan (2016), state that: “Magnitude is a property that every real number has, including integers and fractions. Just as understanding magnitude is a crucial feature of understanding numbers, it is also critical to understanding fractions." When estimating, ordering and reflecting on the result of an operation, and thus evaluating the reasonableness of the results found, essentially it is considering the magnitude (POWELL; ALI, 2018).

When thinking about magnitude, fractions are recognized as numbers, but fractions can also be thought of other meanings, such as part-whole relationships, proportions, quotients, measures or operators (KIEREN, 1980). The importance of magnitude for understanding fractions, as well as its relationship with flexibility and reasonableness, are characteristics listed by Powell and Ali (2018) as defining the understanding of fractions. The authors relate the understanding of fractions into three categories that overlap and interact: flexibility, reasonableness and magnitude, both non-symbolic and symbolic, these categories being a synthesis of numerical understanding in general, and from which the understanding of fractions is a subset.

According to the authors, flexibility refers to conceptions, representations and calculation strategies, resulting in the ability to work with fractions understood in their different meanings. It also includes symbolic and non-symbolic fractions, enabling a change of representation from the written form and visualization of these fractions to the one considered most appropriate. According to Souza and Powell (2021, p. 82), "representative flexibility is characterized by the mental ingenuity of connecting different interpretations of fractions: part-whole, quotient, ratio, operator, measure and the act of measuring". The authors clarify that the part-whole, quotient, ratio, operator and measure interpretations, both for Kieren (1976) and for Behr et al. (1983), are based on the notion of partition of a quantity, while Powell (2019b) presents the notion of fraction from the perspective of measurement, with the fraction being a multiplicative comparison between two quantities.

Reasonableness results in the evaluation of results when operating, approximating or comparing fractions, with the recognition of equivalent fractions, in understanding the correct placement of a fraction on the number line, in reflecting on how an operation involving fractions can change a number, or even what the consequence of performing an operation with a fractional number, which is smaller, greater or equal to 1. In this way, "once the calculation has been performed, the reasonableness of the result would be considered. As a continuation of the fractional sense expression, reasonableness calls plausibility to the question" (POWELL; ALI, 2018, p. 236). Therefore, there is no limit between flexibility and reasonableness, because "their senses inform each other, they also facilitate the apprehension of the concept of magnitude" (SOUZA; POWELL, 2021, p. 85), and the understanding of magnitude is the central concept of the fractional sense (Figure 1).

Figure 1 – The relation between the three componentes of fractions understanding



Source: Powell & Ali (2018, p. 237)

Additionally, Siegler (2016) states that the understanding of numerical magnitudes adds knowledge, both of fractions and whole numbers. According to the author, "for both integers and rational numbers, knowledge of numerical magnitudes is correlated with predictive of, and incidentally related to, other crucial aspects of mathematics, including the achievement of arithmetic and general mathematics" (SIEGLER, 2016, p. 341, our translation). For the author, "numerical development involves the understanding that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines" (SIEGLER *et al.*, 2011, p. 274, our translation). In this way, understanding fraction magnitudes can be a critical step towards a deeper understanding of number.

In this context, Booth and Newton (2012, p. 248) consider, citing Siegler, et al. (2011), that "the lack of understanding of the magnitude of the fraction is rampant among school-age children, but the knowledge of the magnitude of the fraction predicts fraction calculation ability and overall math performance".

Siegler *et al.* (2013) point out three causes for students' difficulty with fractions. The first would be the erroneous assumption that the properties of integers are suitable for all numbers. The second would be the confusion with fraction arithmetic procedures, because, for example, in the addition and subtraction of fractions with the same denominator, it is kept in the result, which is not the case with multiplication and division. And the third cause of difficulty is that children receive instruction in fractions almost exclusively from a part-whole perspective. For the authors, one of the problems of emphasizing this single interpretation can be described in a student's explanation: "You cannot have four parts of an object that is divided into three parts" (SIEGLER *et al.*, 2013, p.13).

As an alternative, Powell (2019b) proposes a conception that is consistent with the historical emergence of fractions through measurement, using Cuisenaire rods. The author carried out the research with the objective of understanding the potentialities of the measurement and fraction-of-quantity perspective to broaden conceptual understandings of the magnitude of fractions. The results of the empirical investigation indicated that “the participants appropriated the idea of magnitude of fractions-of-quantity based on the evoked images of the rods and, by themselves, building expressions of fractional comparisons” (POWELL, 2019b, p. 50). The measurement perspective used by Powell (2019b) refers to the origins of fractions, based on the understanding of the human social practice of comparing or measuring continuous quantities.

In this context, Powell (2018b) presents a pedagogical tool called 4A-Instructional Model. This model consists of four phases of implementation of a “pedagogical approach, the subordination of mathematics teaching to student learning, using Cuisenaire rods” (POWELL, 2018b, p. 409-410)

Chart 1 - *Four phases of our instructional model containing 13 potential instructional activities for implementing thesubordination of the teaching of mathematics to students' learning, using Cuisenaire rods.*

ActualActions	1. Engage the motor and mental powers of students (manipulate, observe, listen, see,hear, abstract, compare, sequence, stress and ignore…). Instruct them to manipulate rods in particular ways so that through their actions on the rods they perceive target relations among the rods. 2. Introduce mathematical language, comparing it, if necessary, to the non- mathematical language that students use, and provide students with opportunities to practice talking mathematically about what they actually perform and perceive with the rods. 3. Have students create their own rod situations that correspond to what is being worked on. 4. Have students talk, draw, and write about what they learn and provide opportunities for practice.
Virtual Actions	5. Engage students in virtual action: manipulating mental images of the rods in wayslike what students performed in actual action. 6. Have students create without rods their own mathematical situations thatcorrespond to what is being worked on. 7. Have students talk and write about what they learn and provide opportunities for practice.
Actions Written	8. Introduce writing mathematical expressions and equations that represent what students can already perform orally and virtually and provide opportunities for practice with the rods available. 9. Have students create expressions or equations with or without rods available. 10. Have students talk and write about what they learn and provide opportunities for practice.
Actions Formalized	11. Formalize symbolically or as a definition the mathematical ideas, concepts, and procedures that have been the basis of students' actual and virtual mathematical manipulations with the rods. 12. Have students talk and write about their understanding of their mathematical ideas in formal, symbolic or definitional statements. 13. Provide opportunities for students to practice their formalized, symbolic or definitional rendition of what they have done with rods.

Source: Powell, 2018b, p. 410.

According to Powell (2018b), the criterion for changing phases is the student's agility on manipulative actions, verbal and symbolic language, mental actions and ideas of an instructional phase. For example, to move from Concrete Actions to Virtual Actions, students need to be comfortable with manipulating the rods, talking about what they are doing and understanding, being able to almost perform actions without the rods.

The Formal Actions phase is the maximum point of the Model. In this phase, the mathematical ideas built by the students in the three previous phases are discussed and written using formal and symbolic language, using algorithms (POWELL, 2018b). When this level is reached, “the algorithm is experienced as an encapsulation or symbolic trace of significant mathematical actions” (SCHMITTAU, 2003, p. 230).

The Cuisenaire Rods

According to Scheffer and Powell (2020, p. 485), Cuisenaire rods constitute a material that “stimulates the evidence and manifestations of understandings in spoken and written form, which contributes to the formalization of mathematical meanings and concepts”. For Gattegno (1960/2009, p. 53), the Cuisenaire material

> provides the child with the means for this playful activity, and it provides all sorts of interesting games (in the commonly accepted sense of the word) through which the skills he will need can be acquired. [...] This is the truly fruitful approach - not the approach in which these elements of experience are made to serve as the means of learning arithmetic.

This material can be used to support a measurement perspective for the knowledge of fraction, as a particular relation of quantities (Gattegno, 1974/2010). On this point, Gattegno (1974/2010, p. 196, author's emphasis), with reference to Cuisenaire rods, summarizes the role of measurement in elementary mathematics:

> Measurement, in working with rods, is borrowed from Physics and introduces counting through the back door, as it is necessary to know how many times the unit has been used to associate a number with a certain length. But measurement is also the source of fractions and mixed numbers and serves later to introduce real numbers. Thus, measurement is a more powerful tool than counting, which is used as a math generator. Count... can be interpreted again as a measure with white rods. Measure is also naturally an interpretation of iteration [...].

Cuisenaire rods have many attributes. One of the attributes is color and another is length (Figure 2), which is used to measure. Due to their simplicity, while students work on mathematical tasks, Cuisenaire rods do not generate a high cognitive load, as they focus their attention mainly on the relationships between the rods, trying to imagine representations for thinking about fractions, relationships between fractions, and operations on fractions (POWELL, 2018b).

Figure 2 – Cuisenaire Rods

Source: Powell (2018b, p. 704).

With Cuisenaire rods, students can build mathematical meanings and mental images by recognizing magnitude, order, equality and inequality, and performing operations with fractional numbers (POWELL, 2018b). According to Gattegno (1976), Cuisenaire rods are probably the best physical material for studying fractions. However, in the last decades, digital technologies have emerged and established themselves, being used for information and communication, and have proved to be extremely important in several areas, including Education. This was accentuated from March 2020, due to the COVID-19 pandemic caused by the new coronavirus — Sars-Cov-2. Schools, teachers and students had to adapt, or even reinvent themselves so that classes were not completely interrupted. In this way, face-to-face classes became remote classes, and face-to-face meetings were replaced by synchronous meetings through video calls and instant communicators.

To carry out this research, secondary data collected in 2020 from the use of digital technologies were used, both for the organization and development of classes (*WhatsApp*, *Google Meet*, *Google Drive*), as a means of teaching and learning (*Applets*[2]), among them, Cuisenaire Digital[3].

[2] Applet is an application that runs inside a larger website or program. They can be animations, simulators, games, among others, which do not require installation on the device used.
[3] Applet with Cuisenaire Rods avaliable in: https://nrich.maths.org/cuisenaire/responsive.html.

Context and Methodological Assumptions

Due to the instability of the global pandemic scenario and the sudden and untimely changes in the organization and realization of classes since March 2020, when face-to-face teaching was suspended, we used secondary data in this research, defined by Malhotra (2004) as those collected for different purposes of the problem at hand, while primary data are those originated by the researcher to solve the research problem.

From the same point of view, Mattar (1996, p. 134), argues that "[...] secondary data are those that have already been collected, tabulated, ordered and, sometimes, even analyzed, with purposes other than meeting to the needs of ongoing research". The secondary data present in this research are the recording of 19 classes planned and carried out in the Remote Emergency Teaching, by the Google Meet platform, with 22 students from the 6th year of Elementary School, distributed in 4 groups, from September 15 to October 16, 2020.

Classes were planned using assumptions of Exploratory Mathematics Teaching (EMT)[3]. Thus, the exploratory tasks were structured considering four phases: task introduction (TI), task completion (RT), collective discussion of the task (DCT) and systematization of mathematical learning (SAM), acronyms that we used during the analysis. The IT and RT phases were performed in one meeting; and in another, the DCT and SAM phases. TI was performed with each group separately. The DCT and SAM phases were carried out with all students in another meeting/class, with at least one day of rest between the first two phases. Table 2 presents the tasks developed, their objectives and the dates of development of the phases of each task, as well as the recording time.

Chart 2 - Tasks of exploratory nature and their objectives, dates and data collection time

Task	Objective	Days of development and minutes		
		IT – RT		DCT – SAM
Task 1: What's the length?	- Understanding fractions as measurement	09/15/2020 G1: 124 min G3: 99 min	09/16/2020 G4: 100 min G5: 61 min	09/18/2020 91 min
Task 2: Measuring with Rods *Cuisenaire* (Part 1)	Understand: equivalence relations and represent them algebraically; fraction equivalence; fractional representation.	09/22/2020 G1: 107 min G3:87 min	09/23/2020 G4: 97 min G5: 80 min	09/25/2020 66 min
Task 2: Measuring with Rods *Cuisenaire* (Part 2)	- Compare fractions; - understand addition of fractions with like denominators.	09/29/2020 G1: 112 min G3: 75 min	09/30/2020 G4: 105 min G5: 82 min	10/02/2020 112 min
Task 3: Train Game	- Compare fractions; - understand addition and subtraction of fractions with different denominators.	10/14/2020 G1:95 min	10/14/2020 G4: 96 min G5:53 min	16/10/2020 76 min

Source: Doneda de Oliveira & Basniak (2021, p. 318).

To organize the meetings and form the groups, the students filled out a form on *Google Forms* to check the availability of days and times, internet access (phone data, wifi) and available artifact (cell phone, tablet, computer with speakers and microphone, laptop). With this information, six groups were formed, with five students in each one. The researcher teacher also created a WhatsApp group for each group, with the aim of: 1) sharing information and communicating with the students before classes; 2) guidance regarding studies and assistance with technological artifacts to those who needed it, since the students were between 10 and 11 years old; and 3) as a way to ask the researcher teacher for help in the task development phase, because although it was connected all the time, it turned off the audio and the camera so that the students could be more self-sufficient.

After the beginning of the task development, with the dropout of some students, there were four groups (G1, G3, G4 and G5) of three to six students, called by pseudonyms chosen by them (Table 3).

Chart 3 – Codinome dos participantes

Groups	Participants
G1	Docinho, Doguinha, Florzinha, Fofinha, Lindinha e Thor
G3	Boom, Flora, Maluquinha, Mazarect e Olívia
G4	Anubis, Bob, Fifo, Magrão, Spider-man e Ymercurius
G5	Luffy, Mulher Maravilha, Poster, Viúva Negra, Zorro

Source: Doneda de Oliveira & Basniak (2021, p. 318).

The data collected come from the recordings and transcripts of the 19 meetings held via *Google Meet*, totaling 1,718 minutes of recordings, approximately 29 hours. In the qualitative analysis, we receive unstructured data to then structure and interpret them (Sampieri *et al.*, 2013). However, according to Sampieri, Collado and Lucio (2013, p. 447), the analysis of qualitative research data is not standard, “because each study requires a scheme or 'choreography' of its own analysis”. In this way, when watching the recordings, the video was paused in the scenes/students' speeches when we identified mathematical expressions referring to mathematical activities resulted of the tasks and the questions of the teacher and colleagues. These statements were located in the transcripts, along with the screen prints and written records of the students.

Therefore, this study is qualitative and interpretative (Creswell, 2010), and its analyzes are based on the phase frame of the 4A-Instructional model (Table 1) and on the differences in the properties of natural numbers and fractional numbers (Table 4) (POWELL, 2019b). We emphasize that the practice was not guided *a priori* by the 4A-Instructional model, but we sought to identify which activities are mobilized by the students in these contexts of practice, aiming at the understanding of fractions.

The focus given to the teaching of fractions in the part-whole meaning tends to lead students to the error in understanding that the set of rational numbers is an extension of the set of natural numbers (Table 4), thus bringing difficulties in conceptual knowledge and procedural of fractional numbers (POWELL, 2018a; 2019b). Therefore, knowing and discussing the properties and differences between natural and rational numbers makes it possible to reflect on the numbers, operations, and results that are produced (magnitude, flexibility, and reasonableness).

Chart 4 – Differences in the properties of natural numbers and fractional numbers

Category of property	Properties	
	Natural Numbers	**Fractional Numbers**
(1) Numerical magnitude signaling	More digits the greater the magnitude: $123 > 23.$ Bigger numeral, bigger number: $9 > 3.$	Neither the number of digits in the numerator or denominator nor the magnitudes of the digits determine the magnitude of the fraction: $\frac{8}{9} < \frac{432}{123}$. Higher value numerals do not indicate a higher number: $\frac{99}{100} < \frac{4}{3}$.
(2) Simbolic representation	Not using operations, magnitude has a single symbolic representation: the magnitude of a set of three items is uniquely symbolized with the number 3.	Not using operations, the magnitude of a comparison between two quantities has an infinity of symbolic representations: one and a half items = $\frac{3}{2} = \frac{6}{4} = \frac{9}{6} = \cdots$
(3) Density	Each natural number has only one immediate predecessor, one immediate successor, or both: For 5, the immediate predecessor is 4 and the immediate successor is 6. Between any two Natural Numbers, the number of Natural Numbers is finite: between 2 and 7 there are four Natural Numbers, 3,4,5 and 6.	Each fraction has no immediate predecessor or immediate successor. Between any two fractions there are infinitely many other fractions. $\frac{2}{7}$ is not the immediate predecessor of $\frac{3}{7}$ Between the two fractions there is, for example, this infinite set of fractions: $\left\{\frac{5}{14}, \frac{7}{21}, \frac{9}{28}, \frac{11}{35}, \frac{13}{42}, \cdots\right\}$.
(4) Product and Quotient	Multiplying can be defined as repeated: $3 + 3 + 3 + 3 = 4 \times 3.$ Multiplying two Natural Numbers different from 1 or 0 to each other produces a product greater than the factors: $4 \times 3 = 12, 12 > 4$ e $12 > 3.$ Dividing any two Natural Numbers that are different from 1 yields a quotient that is less than the dividend: $12 \div 3 = 4, 4 < 12.$	Multiplication as repeated addition is an insufficient definition: $3 \times \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5}$, but $\frac{3}{7} \times \frac{2}{5}$ no. Multiplying two fractions different from 1 or 0 to each other can produce a product smaller than one of the two factors: $\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}, \frac{1}{3} > \frac{2}{15}$ e $\frac{2}{5} > \frac{2}{15}$ Dividing two fractions different from 1 between them can get a quotient greater than the dividend: $\frac{1}{2} \div \frac{1}{4} = 2, 2 > \frac{1}{2}$.

Source**:** Obersteiner *et al.* (2019), apud Powell (2009b, p. 58).

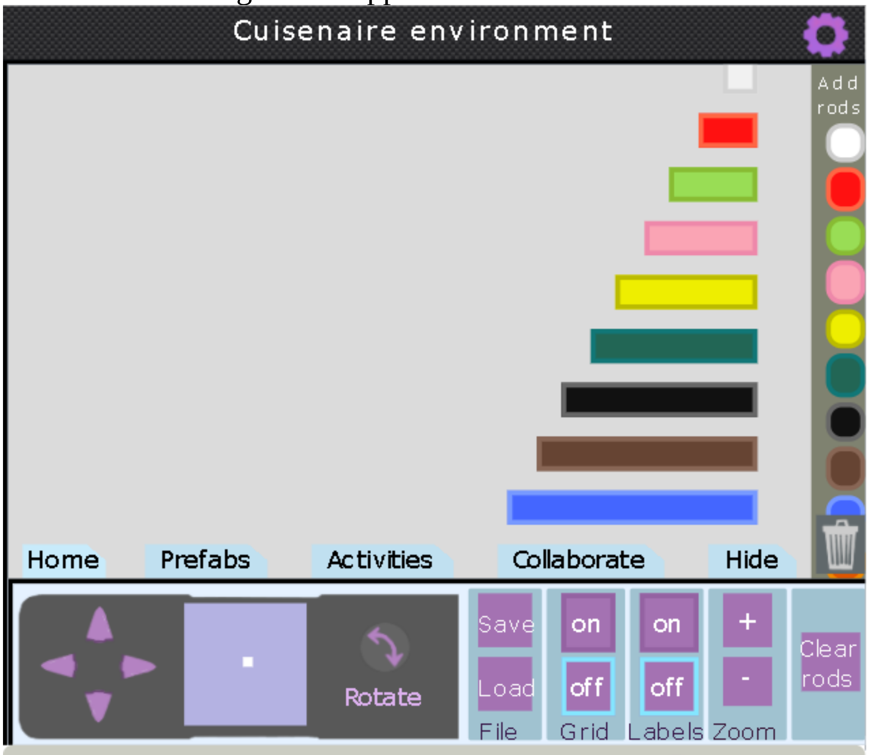
In order to achieve the objective of this article, the analyzes were structured by analyzing the records of the meetings of the IT, RT, DCT and SAM phases (meetings recorded via *Google Meet*, transcripts and written records), searching in the records and, mainly, in the dialogues of students, clues to understanding fractions, including the magnitude of fractional numbers, equivalent fractions, and operations with numbers in fractional form (addition and subtraction). We look for dialogues and indicative records of students' expressions about their mathematical ideas in formal, symbolic or definition statements, in which we identify differentiations and understandings about fractions as a measure, and the properties and differences between Natural Numbers and Rational Numbers; magnitude of fractions, number sense/fractional; in which they recognize that for the same fraction there are an infinity of representations (equivalent fractions) for the same magnitude; in which they identify the differences between operating with Natural Numbers and fractions; introduction to algebra.

We analyze the tasks in the order in which they were proposed and developed (Task 1, Task 2 - part 1 and 2 and Task 3). In addition to the three Tasks mentioned, the initial research included Task 4: Area and Perimeter of Quadrilaterals, in which one of the objectives was to understand the multiplication and division of fractions. This task was not used for our analyses, as other *applets* were used, which are not part of our object of study. We present excerpts from groups G1, G4 and G5, which effectively participated in all tasks and phases of the classes.

Data Analysis

Task 1 had the initial objective of presenting the Cuisenaire rods *applet* and introducing the idea of fractions as measurement. For this, students were challenged to measure the length of the *applet’s* horizontal region (Figure 3). In the IT phase, the teacher asked the students how this could be done, and after the discussions, all the groups came to the conclusion that they could use the rods to measure (no one suggested using a ruler!). When manipulating the rods, we realized that despite being something new, it did not generate a high cognitive load, because, according to Powell (2018b), due to the simplicity of the rods, students focused their attention mainly on acquiring knowledge on the relationships between them.

Figure 3 - Applet Cuisenaire screen

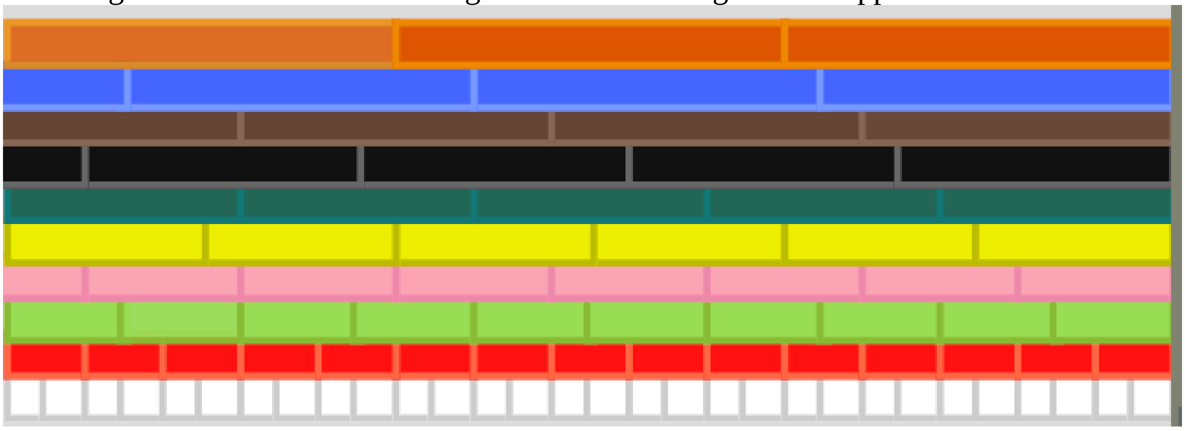


Souce: Research data (2020).

We identified that, when thinking about how to measure and then write the horizontal length of the *applet*, students, in the concrete actions phase of the 4A-Instructional Model, are encouraged to “manipulate, observe, listen, see, listen, abstract, compare, sequence, stress and ignore” (POWELL, 2018b, p. 410), manipulated the rods and perceived relationships between them. The students observed that some lengths of the rods can be composed of lengths of other rods of the same color, involved in multiplicative comparisons.

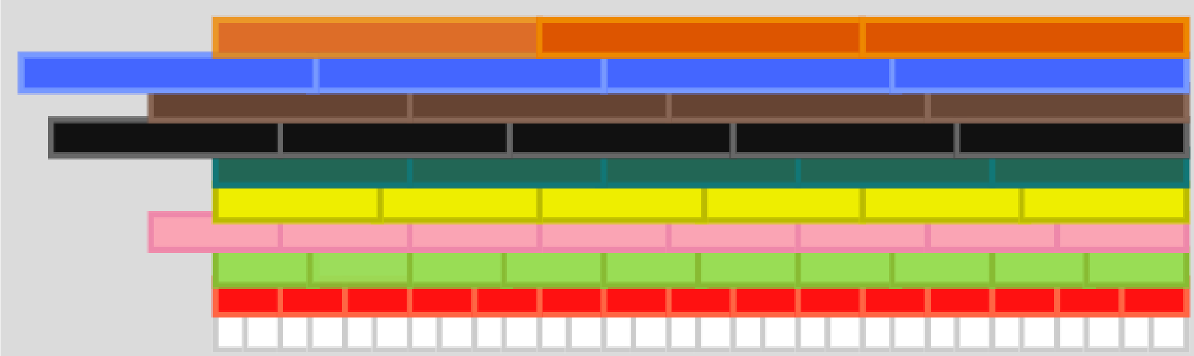
Groups G1, G3, and G5, when measuring the length of the applet, used 30 white bars, which were used as a measurement reference for the other rods. These groups found whole rods that completed the length and others that were missing or left over, in the case of using one more whole rod, as we can see in Figures 5 and 6.

Figure 4 - G5 screen indicating the horizontal length of the applet with the rods



Source: Research data (2020).

Figure 5 - G5 reduced screen indicating the remaining rods



Source: Research data pesquisa (2020).

The G4 used 31 rods, which ended up causing all the other rods to be incomplete, causing some misunderstandings, and requiring the intervention of the teacher. However, in the DCT phase, this brought rich contributions, as it enabled the discussion with concrete examples of concepts, such as prime numbers, multiples, and divisors, as can be read in the excerpt below. This discussion was resumed in Task 3, when students wrote the equivalent fractions requested in the task and found bars with more equivalent fractions than others. Here's the Poster's response:

> **Poster**: *Well, as the horizontal length is 30 blocks and these were complete, you could say they were the multiple numbers of them.*
> **Teacher:** *Oh! multiples. What do you mean by multiple?*
> **Poster:** *Even some of the dividers, I can tell.*
> **Teacher:** *So, give me an example. For example, the white ones, you got 30, right… And where are you seeing multiple and divisor?*
> **Poster:** *In the multiple you can tell by the orange because analyzing it, the orange ones only have three bars and 3×10 is 30.*
> **Teacher**: *Who knows why only the white bar was left whole and the rest was not?*
> **Luffy:** *Because 31 is a prime number.*
> **Teacher:** *Do you agree?*
> **Florzinha**: *I do.*

Task 1 – DCT, 18/09/2020.

When recording the horizontal length of the applet, students used numbers in decimal and percentage form. No group mentioned fraction in task 1, requiring the intervention of the teacher. However, we identified, in the students' expressions, read in the following excerpt, that, when manipulating the rods to write the horizontal length of the applet, the students use the middle (0.5) as a reference to write the length.

Lindinha: *Yes, teacher, I wrote 3.90 here of blue rods; 4.40 of black rods; 7.70 of pink; and that's it, the rest is all exact numbers.*
Teacher: *And that you got to this with some measurement basis, or how did you do it?*
Lindinha: *Like, we imagined it here, with a black rod, 4.40. As it's less than half, so we thought 'ah, since it's almost half, it could be 4.40 or 4.45; at 7.70 it was more than half, and it was a little less than the whole, so we thought it could be 7.80 or 7.65, and then we went to 7.70.*

Task 1 – G1, DCT, 18/09/2020.

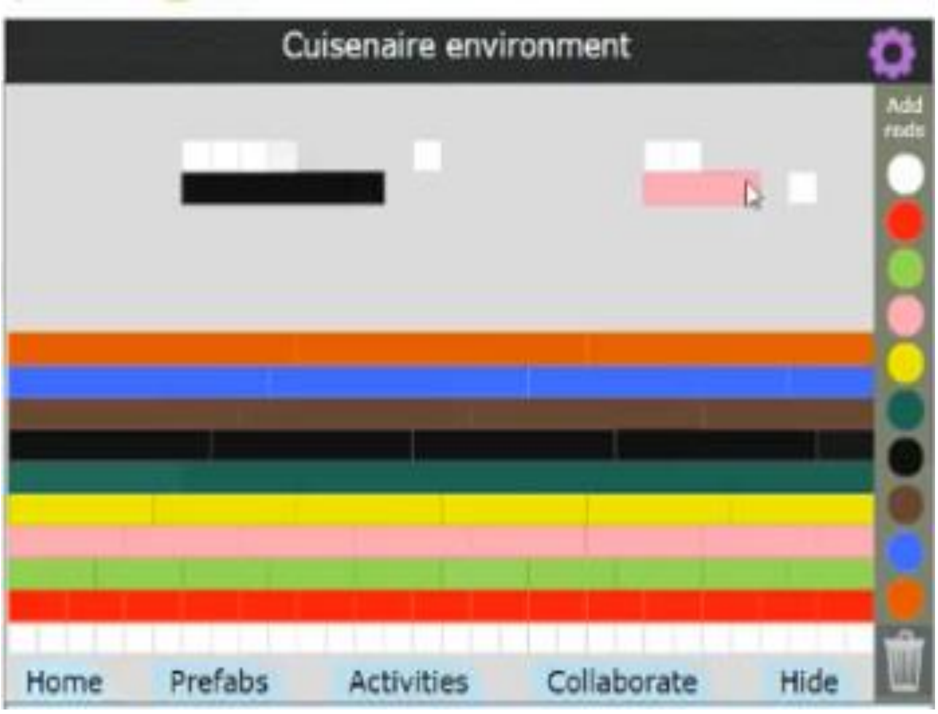
The *medium* would be the *anchor*, a term used by researchers who investigate the number sense (CORSO; DORNELES, 2010; YANG, 2003), understood as a basis for reasoning during the process of solving mathematical problems. Luffy (G5) states that the black rod measures mede 4,25, and when asked by the teacher about what 0,25 would be, the student demonstrates he has important elements of number sense discussed by Yang, Hsu and Huang (2004), such as understanding the meaning of numbers, recognize the magnitude of numbers and use anchor points appropriately. In the following excerpt, student Luffy moves between the decimal and percentage form of rational numbers, using the middle and 50% as anchors.

Teacher: *And the black one?*
Viúva Negra: *4 e a half.*
Teacher: *and a half. Does everybody here agree with half?*
Luffy: *Would be 4,25?*
Teacher: *What do you understand by half?*
Luffy: *50%.*
Teacher: *And what would be 0,25?*
Luffy: *would be half of half.*

Task 1 - G5, RT, 16/09/2020.

After being provoked by the teacher to write the applet length using fractions, Luffy initially tries to find the measurement by comparing the black and the white rods, as it can be seen in Figure 6.

Figure 6 - Black and white rod comparison



Source: Research data (2020).

However, Poster suggests, to make it easier, to use the pink rod. So, after some thought, Luffy agrees and concludes that the applet length measurement using the black rod would be 4 integers and $\frac{2}{7}$.. The measure found is close to the estimated measure using numbers in decimal form, as 2/7 is approximately 0,28, close to the 0,25 that was estimated, demonstrating reasonableness and flexibility as well, by representing a non-symbolic fraction for the form written (symbolic), considering the most appropriate form. We noticed that, using the Cuisenaire rods, the G5, for example, it approached the rational numbers in decimal, fractional and percentage forms in a natural way, without the need for the teacher's intervention, thus enabling the discussion that they are different forms of write the same number.

The idea of equivalent fractions emerged as the groups wrote the horizontal length of the applet with the rods, as can be read in the excerpts from G1 and G5.

Luffy*: There is the pink too.*
Poster*: Each pink block will be 4 white blocks.*
Luffy*: Yes. There are 2 remaining here, so it's* $\frac{2}{4}$.
Poster*: Alright. That is, do I have to put half of the rose?*
Luffy*: Or it could be* $\frac{1}{2}$ *or* $\frac{2}{4}$.

Task 1- G5. RT, 16/09/2020.

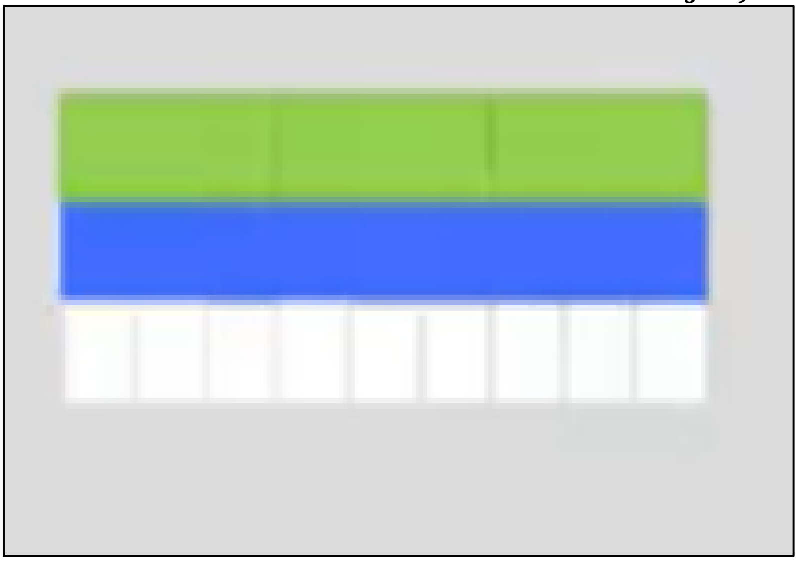
Teacher: *What is the pink of the brown?*
Docinho*:* $\frac{2}{4}$.
Teacher*:* $\frac{4}{8}$. *Or what?*
Docinho*: Half [...] Ok, now, the red one 1V = half of the pink…* $\frac{2}{4}$ *do R [...] E 1V =* $\frac{2}{8}$ *do M and the pink is*$\frac{4}{8}$*, right. Done, teacher.*

Task 2 (part 1) - G1, RT, 09/22/2020.

Luffy*: We made the blue one with 9. Adding 2* [white rods] *to 7, which is black. And you see here there are 3, only and, since there are 9, you can* $\frac{3}{9}$ *or* $\frac{1}{3}$.
Teacher*: And how you got the conclusion it can be* $\frac{1}{3}$*?*
Luffy*: Because* $\frac{1}{3}$ *and* $\frac{3}{9}$ *are the same thing, if you simplify the fraction, that's the same thing.*
Teacher: *And how you can show it to me with the rods, Luffy?*
Luffy*: You just compare it with the white ones.*
Teacher*: Then you told me that* $\frac{3}{9}$ *and* $\frac{1}{3}$ *are equal, how can you show it to me?*
Luffy*: You got to divide 3 by 9 and 3 by 1.*
Teacher*: But is there any way to do it with the rod?* [silence] *you help him, guys.*
Luffy: *Got it. The green is the same length as 3. If you take 3 of these greens, it's 9. And here it would be the same thing, you move it and put it here* [comparing the missing space in the horizontal length with the green rod].

Task 1- G5, RT, 09/16/2020.

Figure 7 - G5's explanation of the equivalence of $\mathbf{\frac{1}{3}}$ e $\mathbf{\frac{3}{9}}$



Source: Reasearch data (2020).

As students manipulate the bars, they realize that it is possible to write more than one expression for the same group of rods, and in this way, the concept of equivalent fractions can be constructed and understood. Because the fractions $\frac{1}{2}$ and $\frac{2}{4}$ or $\frac{1}{3}$ and $\frac{3}{9}$ describe the same relationship between the rods, they are called equivalents. Although recognizing equivalent fractions was not the objective of the task, ideas about equivalence emerged in the RT phase and were addressed in the DCT.

Task 2 - part 1, aimed to understand equivalence relations and represent them algebraically, as well as understand fraction equivalence and represent them symbolically. For this, the groups should initially choose a letter to represent each color of rods. Coincidentally, all groups chose the same letters, as described in Figure 8.

Figure 8 - Letters chosen to para represent the rods

White = B	Red = V	Light Green = C	Pink = R	Yellow = A
Dark Green = E	Black = P	Brown = M	Blue = Z	Orange = L

Source: Research data (2020).

The National Mathematics Advisory Panel (NMAP, 2008) states that a basic knowledge of fractions is crucial for the success of students in Algebra, with the introduction to Algebra being more effectively contemplated in the 7th grade of Elementary School (BRASIL 2018; PARANÁ 2018; 2019), and as the fractions, it is often a very difficult topic for the students. However, when manipulating the rods, using letters to represent them and establishing relationships between their measurements, there was the introduction of mathematical language, which was compared to non-mathematical language by students, when, for example, G5 students said that a light green rod is equal to 1/3 of the blue one, and then they wrote that 1C = 1/3 Z, thus having the opportunity to practice mathematical conversations about what they visualized with the rods and to write expressions and equations , moving from the concrete phase to the written phase (POWELL, 2018b), as we can see in Figure 9.

Figure 9 - Resolution of item b of Task 2 (part 1) by G5

Source: Research data (2020).

The excerpt below refers to when the teacher explains at SAM that the students had worked with algebra.

Teacher*: Let me talk to you about something important. The first thing I want to know is: did you find it very difficult to work with letters?*
Ymercurios*: No.*
Docinho: *Yes, kind of.*
Florzinha*: No.*
Teacher: *t's the first-time you guys work with letters, isn't it?*
Florzinha*: Yes.*
Teacher*: Did you know that many people drop out of school because of this? Did you know that what you just did is called algebra? You replaced the name of the rods with small letters. When we put the letters in mathematics, we are working with the idea of algebra, which you will properly learn there in the 7th grade. Another thing I want to understand, for you to explain to me, what did you do with the rods, only play with them? What was the purpose of this task?*
Poster*: Other than playing, measuring.*

Task 2 (part1) - SAM, 09/25/2020.

According to Booth and Newton (2012) as the rules for operating with fractions can be generalized, fractions can be an excellent way to introduce the use of variables.

In item b of task 2 (part 1), each member of the group should choose a rod and, subsequently, the group would write all combinations of rods of a single color and that were the same size as the chosen rod, forming the maximum of possible combinations. All groups demonstrated that they understood the equivalence relations, representing them algebraically, writing the equivalence them from the largest to the smallest rod, and from the smallest to the largest. The excerpt below, taken from the DCT phase, exemplifies the discussions held by the groups.

Teacher: *So, look, they wrote it with standard writing and then they wrote with small letters. Remember when the teacher says that math looks like a lazy animal? It keeps inventing things to make a shortcut, why? Because it really facilitates this mathematical representation there, right?*
Florzinha*: Yes.*
Teacher: *It saves time, doesn't it? And what did you write forward there, that 1B = to what?*
Docinho*: =* $\frac{1}{2}$ *of V.*
Teacher: *What else?*
Docinho: *1B =* $\frac{1}{5}$ *of A and 1B =* $\frac{1}{10}$ *of L.*
Teacher*: That's it. So first they put the units of measurement as white, orange, yellow, and red; so how big is that in relation to orange? Our unit of measurement was orange,*

then they reversed it. They put white is half of red, white is ⅕ of yellow, and white is $\frac{1}{10}$ of orange, right? And they did it with other rods, the bottom rod is the blue one, isn't it?
Docinho: *Yes.*
Teacher: *You can read it now.*
Docinho*: 9 white rods, 3 light green rods to complete the blue one. 9B=1Z=3C.*
Teacher: *What after?*
Docinho*: 1V =* $\frac{2}{5}$ *of A, 1V =* $\frac{2}{10}$ *of L, 1A =* $\frac{5}{10}$ *of L.*
Teacher: *Yep, those were still in relation to the orange rod, I hadn't even seen it going on. Then you organize it better [...] And the third line is the blue rod too, you can say it.*
Docinho: *1B =* $\frac{1}{3}$ *of C, 1B =* $\frac{1}{9}$ *Z of 1C =* $\frac{3}{9}$ *of Z.*
Teacher*: Ok. We'll see if there's more to come. You can talk about black one.* ***Docinho:*** *7 white rods to complete a black rod. 1B = 1/7 of P.*

Task 2 (part1) - G1, DCT, 09/25/2020.

We identified that the G5 understood magnitude in the DCT phase, when, after presenting the equivalence relations found with the rods that were chosen, Poster explained:

Teacher: *And what do you understand by equivalent?*
Poster: *The equivalent is what it can be, for example, with different numbers, but the value can be, is equal.*

Task 2 (part1) - G5, DCT, 09/25/2020.

Still in the SAM phase, the teacher asked if, when thinking about bar equivalence, would it make any difference to write R=2V, or 2V=R, when Ymercurios answered no. This question refers to the possibility of approaching, with rods, equations, a content that, according to the experience of the first author as a teacher of Basic Education, causes difficulty not only for 7th grade students, when the first degree equations are introduced, but also to high school students. They have difficulty, for example, in recognizing that by inverting the 1st and 2nd members of an equation, the value of the unknown does not change.

Finishing the discussion, in the SAM phase, the teacher addresses the question of the fraction as a measure being represented by two digits (numerator and denominator), when students demonstrate that they understand the equivalent fractions that represent the integer (fractional sense). The teacher asks if anyone still thought that the fraction as a measure represents two numbers and what it means, as shown in the following excerpt.

Teacher: *[...] So, one thing that has to be clear to you, equivalent fractions are fractions that have the same measure. Because of it that 1 =* $\frac{2}{2} = \frac{5}{5} = \frac{10}{10}$ *,*
because they are equivalent, right? [...] Now, another important thing I have to tell you, who thinks a fraction is two numbers?
Ymercurios: *One numer only.*
Teacher: *It's one number only, but what does the fraction one-half mean?*

Ymercurios *Half the number.*

Task 2 (part1) - G4, DCT, 09/25/2020.

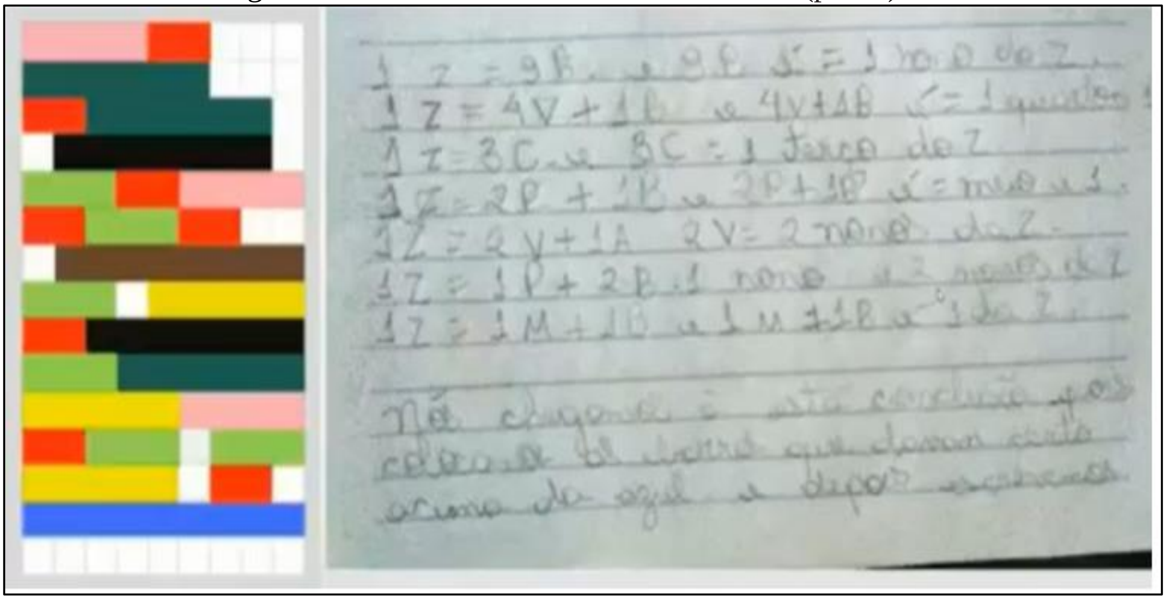
At the beginning of the second part of task 2, the groups should write the combinations of rods of same size, but they could use rods of different colors and, subsequently, compare the fractions found using the symbols of < and >. The task was aimed at adding fractions with equal denominators, establishing additive and comparative relationships with fractions of the same measurement unit. Although it is long, the following excerpt presents the construction of the idea, making it possible to verify if the result makes sense (reasonable) (POWELL; ALI, 2018).

Teacher*: Do you remember that in the past tasks you made a comparison, and we could say, for example, that one rose is equal [to] two red ones.*
Docinho*: Oh! One orange is equal to one blue and one white.*
Teacher*: Very good, and you can also say that one orange is equal to a white plus a blue one, right?*
Docinho*: Yep.*
Teacher: *How the orange relates to the blue and white ones?*
Docinho*:* $\frac{1}{9}$ *?*
Teacher*: So this is another combination that you can put in there. And how do we write this mathematically?*
Docinho: *1L = 1Z and 1B.*
Teacher*: And exchanging 1Z and 1B for a fraction in relation to the orange one, how can we do it?*
Docinho*: 1Z =* $\frac{9}{10}$ *of L and 1B =* $\frac{1}{10}$ *of L.*
Teacher: *Do you agree? [...] And how can we mathematically write that something is orange, for example, without using the orange color. 1= to which fraction?*
Docinho: *=* $\frac{9}{10}$ *of L and* $\frac{1}{10}$ *of B.*
Teacher: *And we can change this and for which mathematical symbol? What are you doing with the blue and white rods to make them the size of the orange one?*
Lindinha: *We are helping.*
Teacher*: And what symbol do we use in mathematics when we are going to put something together?*
Docinho: *Equal? Plus?*
Teacher: *Equal or plus, guys?*
Lindinha: *Plus.*
Teacher: *Do you agree?*
Docinho: *Yes.*
Teacher: *So how can you write using this mathematical symbol?*
Docinho: *That the Orange one is =* $\frac{9}{10}$ *of the blue +* $\frac{1}{10}$ *of B.*
Teacher: *Again, what can we say? To be clear to everybody.*
Lindinha: $1L = \frac{1}{10}$.
Docinho*: No,* $\frac{9}{10}$.

> ***Lindinha:*** *$\frac{9}{10}$... is 1Z + 1B.*
> ***Teacher:*** *Only using fraction, with no using the Orange rod or write L, Z and so?*
> ***Docinho:*** $L = \frac{9}{10} + \frac{1}{10} = \frac{10}{10}$.
>
> Task 2 (part2) – G1, RT, 09/22/2020.

We emphasize that the G5 managed to perform all possible combinations for the yellow rod (16), writing the additive and comparative relations of fractions (Figure 10). The other groups were also able to establish relations, but the chosen rod made it difficult to write all possible ones, as in the case of G1, who chose the orange rod (there would be 512 different combinations).

Figure 10 – Combinations and records task 2 (part 2) G5



Source: Research data (2020).

To accomplish this task, the students compared and operated symbolic fraction-of-quantity expressions using the rods. However, in the DCT phase, when asked by the teacher about the addition and subtraction of some fractions without the use of Cuisenaire rods, the students demonstrated to be in the Formal Actions phase, when ideas are discussed and written using formal and symbolic language, using algorithms (POWELL, 2018b), as shown in the following excerpt.

> ***Teacher:*** *Our goal was to compare fractions and understand addition of fractions with like denominators. What Luffy just came up with is sum of fractions with like denominators, and what you guys also did in the group, so $\frac{1}{5}$+ $\frac{2}{5}$ is how many fifths?*
> ***Luffy:*** $\frac{3}{5}$.
> ***Teacher:*** *Do you understand that? For example, $\frac{2}{3} + \frac{5}{3}$ how many thirds will we have?*
> ***Docinho:*** $\frac{7}{3}$.

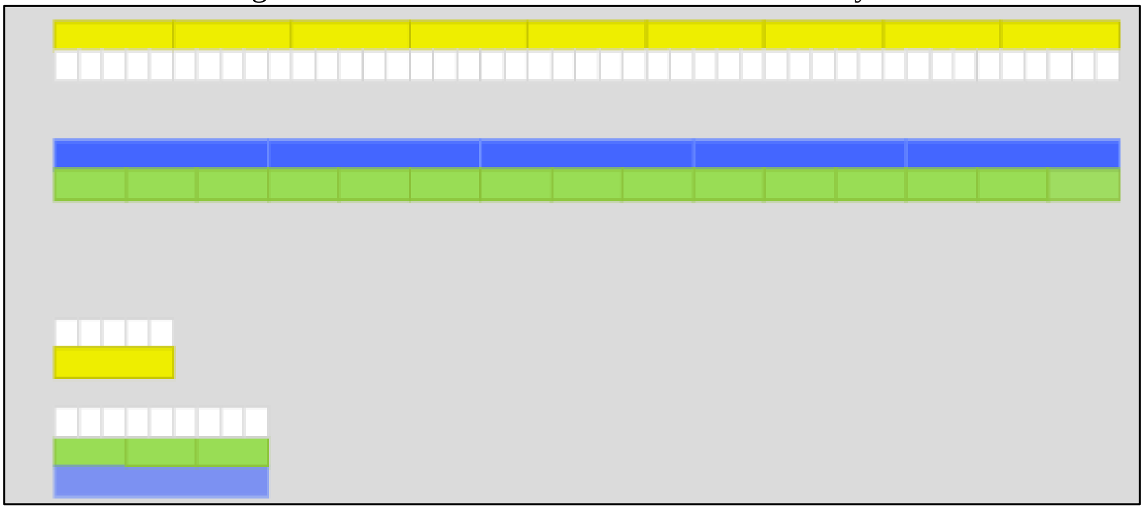
Teacher*: That's it,* $\frac{7}{3}$.
Docinho: *Teacher, so we add only the top part?*
Teacher: *Despite being a single number, yes, but now you know why, right? You know that it's not just a matter of adding the top one because that is what really happens here, if we take the eighth [brown rod], if I take one eighth plus two eighths, how many eighths do we have?*
Luffy: $\frac{3}{8}$.
Teacher*: But now you understand that it's not just a matter of adding with the top one, it's because the parts are the same, we can convert everything to the same unit of measurement, which in relation to the brown, we get the white unit, which is* $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$*, which is three-eighths. So that's why, it's like I just added the top? Yeah, and that works, folks, not only for Cuisenaire bars, if you're going to get* $\frac{3}{15}+\frac{7}{15}$*, is there any way? how much?*
Luffy: $\frac{10}{15}$.
Teacher*: This is because our unit of measurement has not changed, so we can add. And in subtraction, is it the same thing? If I take, like,* $\frac{3}{8}-\frac{1}{8}$ *, how much is that?*
Docinho: $\frac{2}{8}$.

Task 2 (part2), DCT, 10/02/2020.

Task 3 aimed to compare and understand the addition and subtraction of fractions of different measurement units (different denominators). The task started with a game using the Cuisenaire rods, the train game. It was played in pairs, and each player chose a color of rod to be the wagon, and joining the bars (of the same color), they form their train. This game starts with the player who chooses the smallest bar, and the player with the smallest train is entitled to make a new move (no matter how many times). The winning player is the one who, by placing a bar, makes his train the same size as the opponent.

After a few rounds, all groups were able to identify the best strategy to win the game: choosing the smaller rod. To proceed with the task, the groups should choose one of the game rounds and write the fraction that the wagon of each train represented in relation to the whole train, and build trains of the same size and the same color wagon each wagon of each player, then writing fractions equivalent to trains and wagons. G4 chose the yellow and blue wagons. It took 5 blue wagons and 9 yellow wagons to finish the game. Thus, 1/5 represents the fraction of the blue car; and $\frac{1}{9}$, the fraction of the yellow car (Figure 11).

Figure 1 - Resolution of itens b and c of Task 3 by G4



Source: Research data (2020).

G4 and G5 also found no difficulty in adding and subtracting fractions with different denominators after finding the equivalent fractions and choosing those with the same unit of measurement to operate on. However, G1 and G4 needed the teacher's intervention when comparing fractions (magnitude), as expressed in the excerpt below, as they used properties of natural numbers.

***Teacher**: Which one is bigger,* $\frac{1}{5}$ *or* $\frac{1}{9}$*?*
***Spider Man e Ymercurios**:* $\frac{1}{9}$!

After looking at Cuisenaire, they come to the conclusion que $\frac{1}{5}$ is bigger than $\frac{1}{9}$.

***Teacher:** Why **were** you thing that* $\frac{1}{9}$ *é maior?*
***Ymercurios**: **Cause** it's bigger, the numbers! Nine is bigger than five.*

Task 3, RT, G4, 10/14/2020.

The use of rods was extremely important to discuss and clarify the difference between the properties of Fractional and Natural Numbers, as shown in the excerpt.

***Teacher**: [...] Keep going, Docinho.*
***Docinho:** The largest fraction is the blue one, which is 1/4 greater than 1/9, as we cannot say that it is greater by the numbers, regardless of the rods we measure.*
***Teacher:** Explain to the class what you meant by that.*
***Lindinha:** We said that 1/9 was greater than 1/4 because of the numbers, because 9 is bigger than 4, but then the teacher explained that we have to use rods as a measure, and then how we saw that the blue one is bigger than the pink one, hence a 1/4 is bigger than 1/9.*

Task 3 - SAM, 10/16/2020.

G1 chose the pink and blue wagons. It took 4 blue wagons and 9 pink wagons to complete the game. Thus, 1/4 represents the fraction of the blue car; and 1/9, the fraction of the pink wagon. For the pink bar, the group found 1/9 equivalent to 2/8 and 4/36; and for the blue bar, 1/4 is equivalent to 3/12 and 9/36. Finding and understanding equivalent fractions and choosing the appropriate fractions to compare, add and subtract fractions with different denominators, next task items, contributed to the development of fractional sense (flexibility, reasonableness and magnitude). The excerpt below expresses that G1 did not find it difficult to add fractions with different denominators. The result of the operation was also demonstrated by the group with the bars in the DCT (reasonableness) (POWELL; ALI, 2018).

Teacher: *[...] look at the fractions you wrote equivalent to the pink, in item b).*
Docinho*:* $\frac{2}{18}$ *and* $\frac{4}{36}$.
Teacher: *T, what abou the blue?*
Docinho*:* $\frac{3}{12}$ *and* $\frac{9}{36}$.
Teacher*: Where do you have equal unit of measurement? For both pink and blue?*
Teacher*: Exactly. You can do it.*
Docinho: *Teacher, but there it is asking from A [referring to task a].*
Teacher: *But* $\frac{1}{9}$ *is not equivalent to* $\frac{4}{36}$*?*
Docinho*: Yes.*
Teacher: *So, you can do this for both pink and blue and find the same unit of measurement.*
Docinho: *I goes to* $\frac{13}{36}$.
Teacher: *Okay. Now write why you are doing this and why you have this result.*
Docinho: *I'm doing this.*
Lindinha: *Teacher, I didn't understand the question why we are saying this and where we want to go.*
Docinho*: Can I speak, teacher, to be sure that I got it?*
Teacher*: Sure.*
Docinho: *We are adding these two numbers together because they are the same value as* $\frac{1}{9}$ *and* $\frac{1}{4}$*, are equivalents*
Teacher: *Do you understand, Lindinha?*
Lindinha: *Yes.*

Task 3 – G1. DCT, 10/16/2020.

Teacher*: Why did you use tha light green rod too?*
Luffy*: Yes, to measure the blue one. And the fraction* $\frac{1}{2}$ *is bigger than the fraction* $\frac{1}{9}$ *because* $\frac{1}{2}$ *equals to* 50% *of the train, ande* $\frac{1}{9}$ *equals around* 10 *or* 11% *of the train. Now, [n]a d) we transformed the fraction* $\frac{1}{2}$ *into* $\frac{9}{18}$ *and* $\frac{1}{9}$ *in* $\frac{2}{18}$ *for us to be able to add up and reach the conclude that* $\frac{2}{18}+\frac{9}{18}$ *equals to* $\frac{11}{18}$*. Now, [n]a e)* $\frac{1}{2}$ *equals to* $\frac{9}{18}$*, and* $\frac{1}{9}$ *equals to* $\frac{2}{18}$*, and* $\frac{9}{18}$ - $\frac{2}{18}$ *equals to* $\frac{7}{18}$.

Task 3 – G5. DCT, 10/16/2020.

Final Considerations

Although it was not the object of analysis of this study, we found that the fact that the tasks were planned, elaborated, and the classes developed based on exploratory practices, interfered positively with the results achieved. This is because EEM values interaction, involving the teacher and the students in the teaching and learning process. The teaching methodology allowed the students, initially in groups and later in the collective, to manipulate, observe, listen, compare, stress out (as they are used to having straight answers from the teacher), to discuss and then realize the relations between the rods, favoring the construction of mathematical ideas, moving from the Concrete Actions phase to the Formal Actions Phase (POWELL, 2018b).

We identified that, initially, the students did not talk about fractions, using numbers in decimal and percentage form to answer the questions in task 1. However, with the use of rods, they managed to understand that a rational number can be written in fractional form, and not only in decimal and percentage form (flexibility), and although the fraction as measurement is written on two digits, it represents a single number. In addition, they understood the difference in numerical magnitude from natural numbers to fractional ones, recognizing that they made mistakes when operating with fractions because they used the properties of natural numbers.

Regarding the use of the anchor, considered as a basis for reasoning during the process of solving mathematical problems (CORSO; DORNELES, 2010; YANG, 2003), initially the students used 0.5 and 50% as anchors; later they evolved to understand and use 1/2 and integers (fractional sense). There was also an understanding of equivalent fractions, what means that students understood they can write fractions that have the same magnitude (same measure) through different symbolic representations, and use the most appropriate ones to add and subtract fractions with different measurement units (different denominators).

Additionally, we verified that the algebraic language was introduced without generating any cognitive load. In other words, the students used letters to represent the rods and write expressions and equations to establish relationships between their measurements naturally, revealing that the mathematical activity with the Cuisenaire rods constituted an important resource for the introduction of algebraic language and/or to other topics related to Algebra.

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