Onto-semiotic Approach to the Philosophy of Educational Mathematics

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Abstract

In this article, I elaborate on the construct educational mathematics as an ecological variety of mathematics that studies the articulation of formal and applied mathematics, considering the educational contexts. After presenting a synthesis of the main current trends in the philosophy of mathematics, I analyse the contributions of the onto-semiotic approach (OSA) to address the epistemological, ontological, and semiotic problems of educational mathematics. The onto-semiotic configuration of operative and discursive practices, the typology of mathematical objects and processes, and the dualities from which practices and objects can be analysed, provide the essential elements of a specific philosophy of educational mathematics. I articulate an empiricist-factual position in this philosophy for the applied dimension, with a fictional-conventional view of the formal dimension, which helps understand and avoid the educational problems linked to Platonism and physicalism in the teaching and learning of mathematics. Likewise, the educational context requires adopting a transdisciplinary point of view that allows relating philosophical, psychological, socio-cultural, and pedagogical issues, to address the problems of learning and disseminating mathematical knowledge. Finally, I present the ecology of systemic-pragmatic meanings as an essential metaphor of educational mathematics and a synthesis of the OSA philosophical postulates.

Keywords: Educational Mathematics. Philosophy of Mathematics. Onto-semiotic Approach. Transdisciplinarity.

Enfoque Ontosemiótico de la Filosofía de la Matemática Educativa

Resumen

En este artículo, elaboro el constructo matemática educativa como una variedad ecológica de las matemáticas que estudia la articulación de las matemáticas formales y aplicadas, teniendo en cuenta los contextos educativos. Tras presentar una síntesis de las principales corrientes sobre filosofía de las matemáticas analizo los aportes del enfoque ontosemiótico (EOS) para abordar los problemas epistemológicos, ontológicos y semióticos de la matemática educativa. El constructo configuración ontosemiótica de prácticas operativas y discursivas, la tipología de objetos y procesos matemáticos, así como las dualidades desde las cuales se pueden analizar las prácticas y los objetos aportan los elementos esenciales de una filosofía específica de la matemática educativa. En dicha filosofía se articula una posición empirista-factual para la dimensión aplicada con otra ficcionista-convencional para la dimensión formal, lo cual permite comprender y evitar los problemas educativos ligados al platonismo y fisicalismo en la

enseñanza y aprendizaje de las matemáticas. Así mismo, el contexto educativo requiere adoptar un punto de vista transdisciplinar que permita relacionar las cuestiones filosóficas, con las psicológicas, socioculturales y pedagógicas, a fin de abordar los problemas del aprendizaje y difusión del conocimiento matemático. Finalmente presento la ecología de significados sistémico-pragmáticos como una metáfora esencial de la matemática educativa y una síntesis de los postulados filosóficos del EOS.
Palabras clave: Matemática Educativa. Filosofía de las Matemáticas. Enfoque Ontosemiótico. Transdisciplinariedad.

Abordagem Ontossemiótica da Filosofia da Matemática Educativa

Resumo

Neste artigo, elaboro o conceito de matemática educativa como uma variedade ecológica da matemática que estuda a articulação da matemática formal e aplicada, tendo em conta os contextos educativos. Após apresentar uma síntese das principais correntes da filosofia da matemática, analiso as contribuições da abordagem ontossemiótica (AOS) para enfrentar os problemas epistemológicos, ontológicos e semióticos da matemática educativa. A noção de configuração ontossemiótica das práticas operacionais e discursivas, a tipologia dos objetos e processos matemáticos, bem como as dualidades a partir das quais as práticas e objetos podem ser analisados fornecem os elementos essenciais de uma filosofia específica da matemática educativa. Nesta filosofia, uma posição empírico-factual para a dimensão aplicada é articulada com uma posição ficcional-convencional para a dimensão formal, o que permite compreender e evitar os problemas educativos ligados ao platonismo e ao fisicalismo no ensino e aprendizagem da matemática. Do mesmo modo, o contexto educativo exige a adoção de um ponto de vista transdisciplinar que permita relacionar questões filosóficas, psicológicas, socioculturais e pedagógicas, a fim de abordar os problemas da aprendizagem e da difusão do conhecimento matemático. Finalmente, apresento a ecologia dos significados sistémico-pragmáticos como uma metáfora essencial da matemática educacional e uma síntese dos postulados filosóficos do AOS.
Palavras chave: Matemática Educativa. Filosofia da Matemática. Abordagem Ontossemiótica. Transdisciplinaridade.

Introduction

The philosophical reflection on the foundations of the didactics of mathematics as a scientific and technological discipline is essential to adequately guide research, since it conditions the formulation of central questions and the design of instructional models and resources. Likewise, in order to understand and optimise the teaching and learning of mathematics, we should address epistemological questions about mathematics, as proposed by Fundamental Didactics (GASCÓN, 1998), as well as ontological, semiotic, cognitive, and sociological questions, among others. Clarifying the nature of mathematics is essential for mathematics education, both in the contexts of formal uses of creation and justification of mathematical knowledge, and in its application to solve scientific, technological, and everyday life problems. However, it is insufficient, since studying the processes of learning and disseminating mathematics requires considering psychological, pedagogical, sociological aspects, among others. Consequently, mathematics education[1] should adopt a transdisciplinary perspective (ARBOLEDAS; CASTRILLÓN, 2007; STEINER, 1985).

The diversity of philosophical, psychological, and sociological theories, and the dilemmas between epistemological, ontological, semiotic, and cognitive perspectives, led the Onto-Semiotic Approach (OSA) in mathematics education (GODINO; BATANERO, 1994; GODINO; BATANERO; FONT, 2007) to generate a new vision of mathematical knowledge, adapted to the educational context. The OSA assumes that an informed understanding and intervention of instructional processes requires to deal with empirical problems specific to the psychology and didactic of mathematics, such as how we learn mathematical ideas and how we can help to learn them. These questions must be approached together with other properly philosophical questions, such as: What is the nature of mathematical objects and how do they differ from material objects? How do mathematical objects exist? Does mathematics have any ontological presupposition? What is mathematical truth? What is a mathematical proof?

In this paper, I pose the problem of clarifying the relations between the OSA constructs and postulates on mathematical knowledge, and the predominant trends in the philosophy of mathematics. To achieve this aim, I introduce the construct of *educational mathematics* to distinguish, without separating them, pure and applied mathematics when studying mathematical learning. Recognising the specific characteristics of educational mathematics as an ecological variety of mathematics, in which formal reasoning coexists symbiotically with empirical-intuitive reasoning, is important for understanding learning processes and designing informed educational interventions. I will also show that the construct configuration of practices, objects, and processes (onto-semiotic configuration) introduced in the OSA allows articulating coherently elements of a philosophy of educational mathematics, interwoven with psychology and sociology.

I begin the article by characterising educational mathematics as an ecological variety of mathematics that articulates formal and applied mathematics living in diverse educational

[1] Didactics of mathematics in continental European countries.

contexts. I then synthesise the main schools of philosophy of mathematics on which I will project the onto-semiotic approach to mathematical knowledge. In the next section, I present the onto-semiotic configuration construct of the OSA as a tool for analysing educational mathematics. I end the paper with a synthesis of the OSA philosophical postulates on mathematical knowledge, highlighting its transdisciplinary character.

1. Characterising educational mathematics

To address the problems of teaching and learning mathematics, we should clarify the specific characteristics of pure and applied mathematics, as well as their relationships. This analysis reveals the emergence of educational mathematics as an ecological variety of mathematics. We distinguish between the formal and factual (empirical) dimensions of educational mathematics, not considering them as separate dimensions, but maintaining close symbiotic relationships when we are interested in the processes of generating and learning mathematics. Accordingly, the meaning we attribute to “educational mathematics”[2] is different from its use in the Mexican mathematics education community, where it is considered synonymous with mathematics education or didactics of mathematics. “Educational mathematics is thus a discipline of knowledge whose origin dates back to the second half of the twentieth century and which, in general terms, we could say deals with the study of didactic phenomena linked to mathematical knowledge” (CANTORAL; FARFÁN, 2003, p. 29).

Next, we clarify the characteristics of pure and applied mathematics, relying mainly on Bunge (1985). We can describe contemporary pure mathematics, also called abstract, formal, or axiomatic mathematics (MARQUIS, 2014), as the investigation of problems about conceptual systems or their elements to find patterns that satisfy such objects, which are justified by rigorous proof. Mathematics, as a formal science, uses symbols and constructs, but not empirical or factual objects (facts, things, properties of things, and events). Applied mathematics, on the other hand, consists of the investigation of problems arising in factual science, technology, or humanities, with the help of constructs belonging to pure mathematics. Applied mathematics then differs from pure mathematics by:

[2] Fischer (2006) uses the expression "educational mathematics" to refer to the connections between pure and applied mathematics that must be considered in mathematics education.

- The origin of the problems, which is extra-mathematical in the first case and internal in the second.
- The final referents, which are real things in the case of applied mathematics, and constructs in the other case, and
- The goal, which is to help the non-mathematical discipline in the first case, and to advance pure mathematics in the second.

A problem belongs to formal mathematics when its solution requires formal (i.e., non-empirical) proofs or refutations. Applied mathematics makes use of formal constructs and models, but also of empirical artefacts and constructs. Echeverría (2007) adds education and dissemination to these two contexts as a field of reflection in the philosophy of science since it constitutes a fundamental component of scientific activity.

The study of problems from the extra-mathematical and intra-mathematical worlds is addressed even from the earliest levels of education. For example, learning natural numbers begins with counting problems, and assigning a number to the cardinal of a set of perceivable objects. But this requires the simultaneous learning of the mathematical structure, the sequence of number words and symbols, and the principles of counting, due to the interconnectedness of the formal and the applied in educational mathematics. Applied problems involve factual objects and empirical verifications, which must be differentiated from formal constructs and the conventional rules by which they are operated and justified.

Educational mathematics study not only constructs conceptual objects (such as numbers or triangles), which correspond to pure mathematics, but also factual propositions referring to concrete (real, material) things, such as sizes or dimensions of triangular-shaped objects. In teaching mathematics, it is necessary to assure that students do not confuse mathematical objects with their material or symbolic representations. This issue is irrelevant in the applications of mathematics or in pure mathematics, which deals only with abstract entities. The procedures of justification in educational mathematics are also different, because not only logical and deductive procedures are used, but also analogy, metaphor, induction, and plausible reasoning. Special care is taken to distinguish between empirical justifications and deductions from definitions and postulates.

Summarising, we can say that pure mathematics is an activity[3] whose object/motive is the creation of mathematical models to address the solution of increasingly general problems, for which it develops constructs and theories with progressive levels of abstraction and formalisation. The object/motive of applied mathematics is the solution of specific questions in the empirical sciences, technologies, and social sciences by applying available mathematical models. The object/motive of educational mathematics is the study of the dialectical relationships between pure and applied mathematics, between the processes of creation and application of mathematical knowledge, as they are to be the subject of teaching and learning. Consequently, educational mathematics should not only study the process of abstraction (progressive generalisation, synthesis, and formalisation), but also the inverse process of interpretation (analysis, particularisation, and concretisation) and the dialectical relationships between them.

2. Philosophies of mathematics

The philosophies of mathematics developed over the last twenty-five centuries address issues such as the following:

- Ontology: Questions about the ontological status of mathematical objects.
- Semantics: Questions about the meaning, reference, and truth in mathematics.
- Epistemology: Questions about the nature and sources of mathematical knowledge.
- Methodology: Questions of justification (particularly, proof) and application.

These themes are essential and characteristic of the philosophy of pure, applied, and educational mathematics, although in this last case they are intertwined with other questions concerning learning and teaching in different educational contexts and levels. Table 1 summarises the typical principles of five widely recognised philosophies of mathematics.

Mathematical Platonism can be defined as the conjunction of the following three theses: (a) *existence*: mathematical objects exist, and mathematical sentences and theories provide true descriptions of such objects; (b) *abstraction*: mathematical objects are abstract, i.e., non-spatial-temporal entities; and (c) *independence*: mathematical objects are independent of intelligent

[3] We understand the notion of activity in the sense proposed by the Cultural Historical Activity Theory in its second and third-generation version (ENGËNSTRON, 1987), where it is considered as a unit of analysis whose structure includes six elements: subject, object/motive, instruments, community, rules, and division of labour.

agents and their language, thought, and practices. Moreover, according to Platonists, abstract objects are entirely non-physical, non-mental, non-spatial, non-temporal, and non-causal (LINNEBO, 2009). Empirical realism shares with Platonism the view that mathematics is the description of objects that exist independently of people and the language used to represent them. However, rather than situating objects beyond space and time, empirical realism situates them within a spatial-temporal world. The main perspectives in this respect are physicalism, holistic empiricism, and radical empiricism (FONT; GODINO; GALLARDO, 2013). Ernest (1998) has highlighted the negative consequences that Platonism and mathematical realism, as well as foundationalist and absolutist positions, may imply for mathematics education.

Table 1 – Principles of five philosophies of mathematics

Philosophy	**Math objects**	**Mode of introduction**	**Meaning**	**Truth**	**Math knowledge**	**Math activity**
Platonism	Self-existing ideal and eternal	Discovery	Non-contradiction	Formal	A priori and conceptual	Deductive
Nominalism	Symbols	Convention	Nil	Convention	Nil	Formal manipula- tion of symbols
Intuitionism	Mental cons-tructions	Invention	Reducibility to positive integers	Reducibility to numerical computation	A priori and intuitive	Intuitive and rational
Empiricism	Mental	Discovery	Reference to experience	Empirical	Empirical	Trial and error, rational and empirical
Conceptualist and fictionist materialism	Fictions (classes of brain processes)	Invention and discovery	Conceptual reference and contextual sense	Formal	A priori and conceptual	Abstraction, generalization, formal manipulation, trial and error, analogy, induction and deduction

Source: Bunge (1985, p. 120)

In addition to the five philosophies presented in Table 1, other possible relevant contributions to the philosophy of educational mathematics are the philosophical positions on mathematics of Wittgenstein and Lakatos.

Wittgenstein (1976) was mainly concerned with questions of learning, understanding, inventing, and using elementary mathematical ideas. Wittgenstein’s philosophy of mathematics is situated at the opposite end of the spectrum from Platonic-idealistic and psychological approaches. He poses the challenge of overcoming the dominant Platonism and stop speaking of mathematical objects as ideal entities to be discovered and of mathematical propositions as descriptions of the properties of such objects. Alternatively, he proposes that mathematical propositions should be seen as instruments, as rules for transforming empirical propositions. For example, the theorems of geometry are rules for framing descriptions of shapes and sizes of objects, their spatial relations, and for making inferences about them. Wittgenstein’s view of mathematical language as a tool is also relevant for educational mathematics. He argued that words are tools and clarify their uses in our language games. For example, we should not lose sight of the fact that number-words are instruments for counting and measuring and that the foundations of elementary arithmetic, i.e., the mastery of the series of natural numbers, is based on the training in counting.

Lakatos’ (1976) ideas about mathematics are summarised in the following theses (BUNGE, 1985). First, mathematical research is not essentially different from scientific research since it also involves the formulation of conjectures and the search for counterexamples of them. Second, since one often starts from inaccurate concepts and can make mistakes in proving theorems, one has to adopt a fallibilist epistemology of mathematics. Third, formalism does not faithfully represent the real work of the mathematician, which involves non-deductive procedures. In Bunge’s view, all three theses are reasonable, but they do not constitute a philosophy of mathematics. Lakatos does not express clear ideas about the nature of mathematical objects: he was more interested in the history than in the ontology or semantics of mathematics. Like anyone else trying to solve a problem, the professional mathematician is obliged to use analogy and induction and testing conjectures until he finds the correct solution, even using material tools. However, the logic of mathematical discovery and the heuristic procedures in problem-solving described by Lakatos provide important elements for the philosophy of educational mathematics. The progressive mathematical growth, both from a cognitive and a historical-cultural point of view, need not be linked to a deductivist style but follow in the footsteps of the heuristics described in the book *Proofs and Refutations*. However, this does not mean that pure mathematics, as a specific epistemological formation, is not fundamentally different from the factual sciences.

In mathematics education, some authors address issues related to the philosophy of educational mathematics. Such is the case of Sfard (2000), when she analyses the relationships between symbols and mathematical objects. The problem addressed by her, expressed in semiotic terms, is: “mathematical symbols refer to something – but to what? … What is the ontological status of these entities? Where do they come from? How can one get hold of them (or construct them)? (SFARD, 2000, p. 43). Sfard rejects the conception that conceives signs as independent of meanings and adopts the view of psychologists (such as Vygotsky) and semioticians (such as Peirce) that signs (language in general) play a constitutive role in objects of thought and not merely a representational role. The central thesis defended by Sfard is that:

> Mathematical discourse and its objects are mutually constitutive: It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions (SFARD, 2000, p. 47).

Font and collaborators (2013) argue that the way in which mathematics is taught in schools leads students to implicitly develop a realist view of mathematical objects. This view assumes that mathematical statements are a description of reality and that the mathematical objects portrayed by these statements are part of this reality.

> In the teaching process this “reality” to which mathematical objects belong is located at an intermediate point between what, in the philosophy of mathematics, are referred to as Platonic and empiricist positions, although depending on the teaching process considered, one may observe a clear preference for one or the other of these two points of view, for example, in contextualized teaching or realist mathematics (FONT; GODINO; GALLARDO., 2013, p. 99).

A complementary explanation for this educational phenomenon is that teachers and mathematics educators do not discriminate the substantial differences between applied and formal mathematics, and that educational mathematics needs to identify the conflicts and obstacles generated in the learning processes when these differences are not considered.

3. Fundamentals of educational mathematics in the OSA

Problematising the type of mathematics studied in education systems is necessary in the OSA. It assumes that educational mathematics must adopt a specific perspective of mathematics adapted to the learning and teaching context. This vision complements the formal-logical view in the contexts of creation and justification of mathematical knowledge, and the empiricist-factual perspective linked to the contexts of application. It is essential to distinguish between pure or formal mathematics, applied mathematics, and educational mathematics, which results from ecological processes of adaptation of mathematics to different contexts and educational levels. Consequently, we need a philosophy of educational mathematics that addresses the epistemological (emergence and development of mathematical knowledge), ontological (nature and types of mathematical objects), and semiotic (syntactic, semantic, and pragmatic) problems specific to this variety of mathematics.

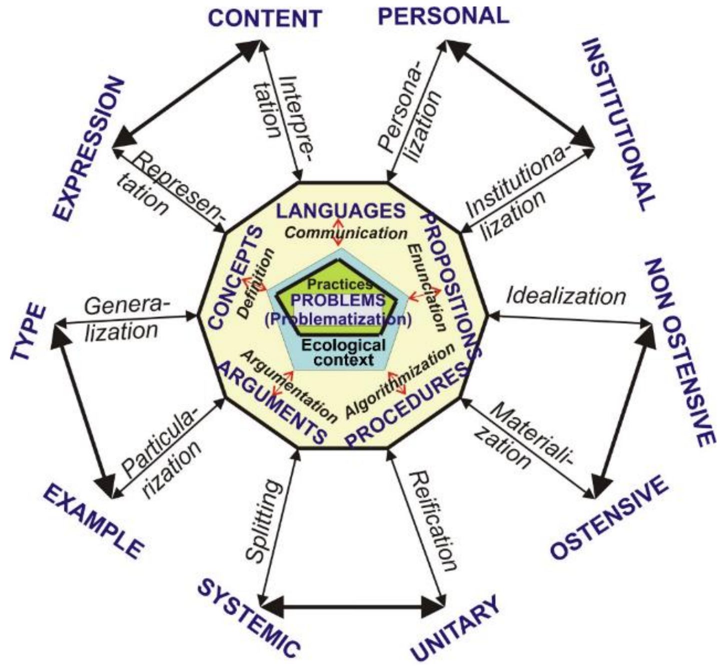
The educational context also requires articulating the philosophical problems of mathematics with questions related to the cognitive processes involved in learning, which takes place in historical-cultural contexts that condition and support them. Therefore, the foundations of mathematics education require the development of specific theoretical models that consider the philosophical, psychological, and socio-cultural issues (among others) involved in the teaching and learning of mathematics.

In the following sections, I present a synthesis of the assumptions and theoretical constructs elaborated in the OSA that shape the essential features of a specific philosophy of educational mathematics. The OSA theoretical constructs that articulate central questions of the philosophy with the psychology and sociology of educational mathematics are:

- Mathematical practices.
- Mathematical objects and processes.
- Contextual attributes of practices and objects.

These theoretical constructs are articulated in the *configuration of practices, objects, and processe*s (onto-semiotic configuration) tool, which we explain in the following sections. The onto-semiotic configuration tool (see Figure 1) incorporates elements of the notions of concept, conception, schema, mathematical praxeology, and semiotic representation register used in mathematics education. In Godino and collaborators (2011), we show the analytical breakdown provided by the onto-semiotic configuration, for both institutional and personal knowledge, with an example related to the concept of natural number. The semiotic function tool (expression-content duality) is also exemplified to analyse the learning of the ten. Likewise, Font and collaborators (2013) study the emergence of mathematical objects based on the practices performed to solve mathematical problems.

Figure 1– Onto-semiotic configuration of practices, objects, and processes



Source: Modified from Godino (2014, p. 23)

3.1. Mathematics as an activity

People's problem-solving activity in a specific ecological (material, biological, and social) context is the central element in constructing mathematical knowledge (see Figure 1). The approach to the epistemological problem of the genesis of knowledge is made operational in the OSA with the notion of *mathematical practice*, which is defined as "any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to others" (GODINO; BATANERO, 1994, p. 334).

To solve a problem, the subject carries out an organised sequence of different types of operative and discursive practices. The answer to the epistemological question of how mathematics emerges and develops is an anthropological (WITTGENSTEIN, 1953) and pragmatist (PEIRCE, 1958) view of mathematics.

The same problem is solved by systems of practices that depend on the ecological context in which they arise, e.g., communities of mathematical practitioners, people developing new mathematical knowledge or applying it, and educational contexts. The practices relativity from institutional, temporal, and material context add a sociological and historical dimension to the epistemology assumed by the OSA.

> An institution is constituted by the people involved in the same class of problem-situations, whose solution implies the carrying out of certain shared social practices and the common use of particular instruments and tools. (GODINO; BATANERO, 1994, p. 336).

The problems that origin or motive mathematical activity can be extra-mathematical (involving things, objects, and material facts) or intra-mathematical (in which non-material or ideal objects of reason take part). In educational mathematics, especially at the first levels, the starting point is extra-mathematical problems related to the environment and everyday life; so, the objects involved in the practices can be material artefacts and abstractions, both empirical and formal or theoretical.

From an educational point of view, it is important to postulate that the mathematical activity involved in learning mathematics is different from the activity of mathematicians who construct new knowledge. In the first case, the learner reconstructs knowledge, which already has a historical-cultural existence, whereas in the second case, new postulates are invented, and new relations derived from previously elaborated knowledge are discovered. From this fact, it follows that the first encounter of students with new types of problems and the mathematical objects created to solve them may require interaction patterns in which collaborative work and the transmission of knowledge predominate over the learner's autonomous work.

Personal-institutional contextual duality

The OSA articulates the epistemic and cognitive facets of mathematical knowledge by attributing to mathematical practices a dual character, namely, personal (individual) or institutional (social). Mathematical practices can be idiosyncratic to an individual or shared within an institution or community of practice. There are no institutions without individuals, nor individuals without the various institutions they are part of (family, school, etc.). The distinction between personal and institutional practices makes it possible to become aware of the dialectical relationships between them; on one hand, individuals are subject to the modes of action shared within the institutions of which they are part of; on the other hand, institutions are open to the initiative and creativity of their members. This postulate links the cognitive (psychological) to the epistemological and sociological dimensions of mathematical knowledge. Besides its logical-formal dimension, another factual dimension of mathematics accounts for the processes of creation of mathematical objects, which emerge from practices, not as Platonic ideal existents that are discovered (FONT; GODINO; GALLARDO, 2013). From a personal point of view, mathematical objects have a mental/neural existence, while from an institutional point of view, they possess a cultural existence.

3.2. Mathematics as a system of objects and processes

Mathematics is not only an activity of individuals but also a system of culturally-shared objects emerging from that activity. We should address the ontological problem, i.e., clarifying what a mathematical object is, what types intervene in mathematical activity, what the mode of being of mathematical objects is, and in what sense mathematics speaks of objects (PARSON, 2008). In the OSA, mathematical practices, that is, the actions performed by people in certain types of problems, are the origin and *raison d'être* of mathematical abstractions, ideas, or objects (FONT; GODINO; GALLARDO, 2013). It is postulated that a mathematical object is any material or immaterial entity that intervenes in mathematical practice, supporting or regulating its realisation. It is a metaphorical use of the term *object,* since a mathematical concept is usually conceived as an ideal or abstract entity and not as something tangible, such as a rock, a drawing, or a manipulative artefact. This general idea of the object, consistent with that proposed in symbolic interactionism (BLUMER, 1982; COBB; BAUERSFELD, 1995), is useful when complemented with a typology of mathematical objects by considering their different roles in mathematical activity.

The six types of primary entities assumed (see Figure 1) extend the traditional distinction between conceptual and procedural entities, which are insufficient to describe the intervening and emergent objects of mathematical activity. *Problems* are the origin or *raison d'être* of mathematical activity; *language* represents the remaining entities and serves as an instrument for actions; *arguments* justify the procedures and propositions that relate concepts to each other. *Concepts* (number, fraction, derivative, etc.), as components of onto-semiotic configurations, are conceived as definitions, a different view from that proposed by Vergnaud (1990) as the triplet formed by situations, operative invariants, and representations. The idea of concept as a system is taken up in the OSA by the onto-semiotic configuration construct. In addition, configurations are organised into more complex entities such as conceptual systems or theories.

The constitution of personal and institutional objects and relations occurs over time through mathematical processes, which are interpreted as sequences of practices. The emerging mathematical objects form the crystallisation or reification of such processes. Interpreting mathematical processes as sequences of practices in correspondence with primary mathematical object types provides criteria for categorising them. The formation of linguistic objects, problems, definitions, propositions, procedures, and arguments takes place through the respective primary mathematical processes of communication, problematisation, definition, enunciation, elaboration of procedures (algorithmisation, routinisation, etc.), and argumentation. Problem-solving – and more generally, modelling, should rather be considered as a mega-process, as it involves the articulation of primary processes (establishment of connections between objects and generalisation of procedures, propositions, and justifications) (GODINO; BATANERO; FONT, 2007).

The personal-institutional duality also applies to objects and processes. When the systems of practices are shared within an institution, emergent objects are considered institutional objects, whereas if such systems correspond to an individual, they are considered as personal objects. Personal objects include cognitive constructs, such as conceptions, schemas, internal representations, etc.

3.3. Mathematics as a system of signs

Educational mathematics must address the following semiotic-cognitive problem: What is it to know and understand a mathematical object? What does an object mean for a subject at a given time and under given circumstances? These questions are analysed in the OSA by considering that mathematical activity and the processes of construction and use of mathematical objects are essentially relational. Different objects are not conceived as isolated entities but related each other. For example, between the symbol 2 and the concept of number 2, as well as between the concept of natural number and the system of operative and discursive practices from which this mathematical object emerges, a relationship is established, which the OSA calls a *semiotic function*. The semiotic function is the correspondence between an antecedent object (expression/signifier) and a consequent object (content/meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. This construct is included in Figure 1 as the expression-content duality, which allows to account for any use of meaning: meaning is the content of a semiotic function (GODINO; BURGOS; GEA, 2021). The OSA assumes that any entity that participates in a semiosis process, interpretation, or language game, is an object and can play the role of expression (signifier), content (signified), or interpreter (a rule that relates expression and content). The systems of operative and discursive practices are also objects and can be components of the semiotic function. The *systemic-pragmatic meaning* of a concept (in general, of any object) is the system of operative and discursive practices performed by a person (personal meaning) or within an institution (institutional meaning) to solve a type of mathematical problem.

The semiotic function makes it possible to describe mathematical knowledge comprehensively as the set of relations that the subject (person or institution) establishes between mathematical objects and practices. Talking about knowledge is equivalent to talking about the content of one (or many) semiotic functions, resulting into a multiplicity of types of knowledge in correspondence with the diversity of semiotic functions that can be established between the various types of practices and objects. Since the system of practices involved in problem-solving are relative to individuals and communities of practices (institutions), pragmatic meanings, and thus knowledge, are relative. However, it is possible to reconstruct a global or holistic meaning of an object by systematically exploring the contexts of use of the object and the systems of practices involved in its solution. Such holistic meaning is used as an epistemological and cognitive reference model of the partial meanings or senses that the object may take (GODINO; BURGOS; GEA, 2021). The constructs of institutional and personal meanings allow to interpret understanding in terms of the ongoing matching of the subject's meanings with the institutional reference meanings (GODINO; BATANERO, 1994).

The OSA assumes that the objects, put into correspondence in semiotic functions are not only ostensive linguistic objects (words, symbols, expressions, diagrams, etc.), but concepts, propositions, procedures, arguments, even problems can also be antecedents of semiotic functions. For example, we can ask about the meaning of the concept of number, or the meaning of propositions, procedures, arguments, situations, and representations involved in numerical practices. The expression (signifier) and content (signified) in the semiotic function can also be unitary or systemic entities, particular or general, material, or immaterial, personal, or institutional. This diversity of objects generates a variety of types of meanings, therefore, of knowledge and understandings, which guides and supports onto-semiotic analyses of mathematical activity at macro and micro levels, both from the socio-epistemic (institutional) and cognitive (personal) points of view (GODINO; BURGOS; GEA, 2021). Thus, in the OSA, cognition is understood as pragmatist but also empiricist and rationalist. Action is the source of knowledge, but also perception and reason are.

3.4. Idealisation, reification, and generalisation in the OSA

Three pairs of contextual attributes have been introduced into the OSA ontology to view practices and primary objects and account for the processes of idealisation, reification, and generalisation: ostensive-non ostensive (material, immaterial), unitary-systemic, and extensive-intensive (particular-general; example-type). These constructs serve to describe the types of abstraction (empirical and formal) into play in mathematical activity and the objects that intervene and emerge in these processes. They also help to understand the interweaving between pure and applied mathematics, between constructs and things, which is necessary for educational mathematics since, in learning processes, at least at the first levels, we start from tangible reality to access the virtual reality of formal mathematics.

Ostensive/non-ostensive duality

In OSA, the adjective ostensive is applied to any object that is public and can therefore be shown directly to another person. Symbols, notations, gestures, graphic representations, and material artefacts have this character; they are real or concrete objects. Concepts, propositions, procedures, and arguments are constructs, creations of the human mind, non-ostensive objects; they depend on subjects, their actions, and artefacts for their existence. Non-ostensive objects can be mental objects (when they intervene in personal practices), or institutional objects (when they appear in shared practices). However, mental processing and interpersonal communication of non-ostensive objects require to materialise them through empirical ostensive representations. In both cases, non-ostensive objects regulate the mathematical activity, while their ostensive representations support or facilitate the performance of such work. The distinction between ostensive and non-ostensive objects is relative to the language game in which they participate. Ostensive objects can also be thought, imagined by a subject, or be implicit in mathematical discourse (for example, the multiplication sign in algebraic notation); in these cases, they function as intensive objects. This duality allows us to describe the dual processes of idealisation and materialisation in mathematical activity.

The unitary-systemic duality

In some circumstances, mathematical objects participate as unitary entities (assumed to be known beforehand), while in others, they intervene as systems that must be decomposed for their analysis. “One and the same object can now be regarded as an individual, now as a set (or as a concrete collection). There is nothing final about being an individual" (BUNGE, 1977, p. 107). For example, in the study of addition and subtraction, in the last levels of primary education, the decimal numbering system (tens, hundreds, etc.) is something known and, consequently, a unitary (elementary) entity. This object, in the first grade, is considered in a systemic way for learning. Both the onto-semiotic configurations (in their socio-epistemic or cognitive version) and the primary objects that compose them can be considered from unitary or systemic perspectives, depending on the language game in which they participate. In the first case, processes of reification (synthesis) take place and, in the second case, decomposition (analysis) of the system into its components.

Extensive-intensive (example-type) duality

A characteristic feature of mathematical activity is the attempt to generalise the types of problems addressed, the solving procedures, definitions, propositions, and justifications. Solutions are organised and justified in progressively more general structures. However, in the instructional processes, one begins to study models of these general structures. The analysis of mathematical activity, therefore, requires considering both the processes of particularisation and generalisation, and the objects involved in them. The process of generalisation entails finding or conjecturing a pattern from similar cases, while particularisation involves generating individual examples that follow a pattern.

The contextual attribute extensive-intensive, applicable to practices and objects, has been introduced in the OSA to analyse the dialectic between particularisation and generalisation. Depending on the situation, an object can be an exemplar (extensive) if it intervenes by itself, or a type (intensive) if it represents a wider class.

> An extensive object is used as a particular case (a specific example, i.e., the function y = 2x + 1), of a more general class (i.e., the family of functions y = mx+n), which is an intensive object. The terms extensive and intensive are suggested by the two ways of defining a set, by extension (an extensive is one of the members of the set) and by intension (all the elements are considered at the same time). By extensive we understand a particularized object (individualized) and by intensive, a class or set of objects. (FONT; CONTRERAS, 2008, p. 169).

Font & Contreras (2008) conduct a microscopic analysis on the objects, processes, and semiotic functions at stake in the definition of the derivative of a function. They apply the ostensive-non-ostensive, extensive-intensive, and expression-content dualities to explain the semiotic conflicts posed by the dialectic between the particular and the general in mathematics education.

3.5. Abstraction processes and abstract objects in the OSA

In a first approximation, the ostensive/non-ostensive duality and the associated processes of materialisation and idealisation account for the concrete (ostensive) and abstract (ideal) objects usually considered in everyday language. But the professional and educational analysis of mathematical activity requires a deeper understanding of the abstraction process, the emerging objects, and the inverse interpretation process. For this reason, the OSA complements the ostensive/non-ostensive duality with the unitary/systemic and example/type dualities. Whereby the mathematical abstract object is not only an ideal (non-ostensive) entity but also a generality, considered as a unitary whole or as a system, depending on the circumstances. As Sinaceur (2014, p. 93) states, “the division abstract/concrete integrates the distinction general/particular and class/individual”. Moreover, since reified objects are symbolically represented to intervene in new practice systems, abstraction, i.e., the generation of mathematical knowledge, it also involves the expression/content duality and the processes of representation and signification.

The postulate of the emergence of the object from practices (actions) undoubtedly requires relating the model of abstraction described here to the reflexive abstraction of Piagetian genetic epistemology. *Reflexive abstraction* is the fundamental genetic process that makes it possible to build a new structure from a previous one and consists of extracting certain elements from lower structures to reflect them in new operations, generalising these elements in a higher structure.

> Reflective abstraction starting from actions does not imply an empiricist interpretation in the psychologist's sense of the term, for the actions in question, are not the particular actions of individual (or psychological) subjects: they are the most general co-ordinations of every system of actions, thus expressing what is common to all subjects, and therefore referring to the universal or epistemic subject and not the individual one. (BETH; PIAGET, 1974, p. 238)

In addition, the mathematical practices from which abstract objects emerge are intentional, in the sense that they are performed to solve problems in specific contexts. Consequently, the mega-process of abstraction and the emerging abstract objects is conditioned and supported by the ecological (material, biological, and social) context in which the activity occurs. This observation leads to relate the OSA view of abstraction with *abstraction in context* (HERSHKOWITZ; SCHWARZ; DREYFUS, 2001).

> A process of abstraction is influenced by the task(s) on which students work; it may capitalize on tools and other artifacts; it depends on the personal histories of students and teachers; and it takes place in a particular social and physical setting. We thus take a sociocultural point of view, as opposed to a purely cognitive or a purely situationist one. (HERSHKOWITZ; SCHWARZ; DREYFUS, 2001, p. 195-6)

However, the analysis of the concordances and complementarities of the OSA abstraction model with reflective abstraction and abstraction in context will have to be addressed in other papers.

3.6. Educational mathematics as ecology of meanings

Toulmin (1977) introduced the expression *intellectual ecology* into the epistemology of knowledge to describe the function and adaptation of concepts and methods of thought to the real needs and demands of problem situations. On his part, Morin (1992) considers inadequate the belief in the physical reality of ideas, as denying a real and objective existence to the habitat, life, customs, and organisation of ideas. For Morin, ideas in general (and, therefore, mathematical notions), besides constituting instruments of knowledge, have their characteristic existence. The locus or place of mathematical reality is, for White (1983), the cultural tradition, i.e., the continuum of behaviour expressed by symbols. Within the mathematical culture body, actions and reactions occur between the elements. “One concept reacts on others; ideas mix, merge, form new syntheses” (WHITE, 1983; p. 274).

The ecological metaphor of ideas is helpful for analysing the relationships between school mathematics and expert mathematics. These relationships are often subordinate, which originated the metaphors of didactic transposition, elementarisation, and transformation (SCHEINER et al., 2022) used to describe the processes of selection and elaboration of school mathematics. In the OSA, the ecology of meanings metaphor is proposed to describe these processes and the relationships between different types of mathematics (GODINO; BATANERO, 1998). Each mathematical object has distinct meanings, with various degrees of generality and levels of formalisation; consequently, educational agents select and sequence the appropriate meanings according to the context, students’ abilities, and motivations. The ecological metaphor well reflects the phenomena of competition, symbiosis, collaboration, and, in a sense, the trophic chains established between different types of mathematical knowledge (GODINO, 1994). Only the best adapted knowledge to the context survives or thrives.

The ecological metaphor of school knowledge assumes that there is no single mathematics but multiple and diverse mathematics, not only as a starting point (professional contexts) but also as a point of arrival (school contexts). The progressive growth of knowledge across the curriculum is better described as a phenomenon of mutation-driven by educational activities, from simpler to more complex forms, than as a phenomenon of transposition or transformation from more abstract to more elementary forms (SCHEINER et al., 2022). The ecology of meanings, understanding the meanings of concepts systemically and pragmatically (GODINO; BURGOS; GEA, 2021), more accurately reflects the correspondences between the different types of knowledge involved in educational settings. Interpreting the meanings of a mathematical object in terms of systems of practices facilitates the consideration of these systems, and consequently pragmatic meanings, as new objects that relate to others to form new structures.

4. An overview of the OSA philosophical postulates

The plurality of paradigms and theories that concur in mathematics education, and the need to clarify and articulate them, are a source of inspiration for the emergence of the OSA as a field of scientific and technological enquiry (GODINO, 2022). Aiming to overcome the boundaries between philosophical, psychological, and sociological disciplines, as far as they are interested in mathematics, learning, and dissemination, we developed the onto-semiotic configuration construct that incorporates transdisciplinary elements, as reasoned in section 4. An essential postulate in the OSA is the emergence of mathematical constructs (concepts, propositions, etc.) from the operative and discursive practices when solving problems (FONT; GODINO; GALLARDO, 2013). Mathematical constructs or ideas do not have an existence independently of people but are simultaneously creations and discoveries (CAÑÓN, 1993), thus assuming an anti-Platonist position. Mathematical axioms and postulates are inventions that take place in people's brains, and although the propositions derived from them are not *a priori* known and give the impression that they are discovered, this does not justify Platonism.

The philosophy of educational mathematics proposed by the OSA, implicitly embodied in the onto-semiotic configuration construct (Section 4), is summarised in the following postulates:

Ontological dimension.

- Naturalism: The OSA admits material existents and rules out the independent existence of ideas, be they empirical or formal abstractions. But it rejects physicalism since it denies that all objects are physical entities. Mathematical practices are people's actions and therefore are cerebral and bodily processes (manipulative and gestural). When these practices are shared within a community, they are institutional practices, which are dependent on the brain activity of their members, and the interpersonal interactions established between them.
- Systemism: The object of study in the OSA are the systems of practices, objects, and processes, and the contexts in which mathematical activity occurs, articulated in the onto-semiotic configuration construct.
- Emergentism: The abstract mathematical object comes from other previous entities (the operative and discursive practices) and is not reducible to them.
- Pluralism: Diversity of practices, objects, and processes required for the description and understanding of mathematical activity in its diverse varieties.
- Dynamism: Meanings change with time and personal and contextual circumstances.

Epistemological dimension

- Realism: Mathematical knowledge, both formal and applied, emerges from the people's operative and discursive practices when solving problems. A kind of virtual or fictional reality is granted to the objects that emerge from mathematical activity in interaction with perceptual objects and artefacts in the environment.
- Evolutionism: Personal and institutional meanings evolve and develop over time as subjects tackle successive, progressively more complex problems. Building new knowledge starts from existing knowledge, expanding and correcting the previously produced by individuals within historical communities.
- Social constructivism: Cognitive onto-semiotic configurations are creations of subjects, and socio-epistemic configurations result from interpersonal communication. Constructing of knowledge takes place by the subject, but in a community, whose norms promote or inhibit investigative activity.

- Rationalism and moderate empiricism: Both reason and experience are necessary for building mathematical knowledge; mathematical practices can be both operative (involving the use of empirical artefacts) and discursive (involving objects of reason).
- Conventionalism: Concepts-definition, propositions, and mathematical procedures are conventional rules, not arbitrary but motivated by the activity of description and explanation of objects and facts of the real world and virtual constructs. This conventional character explains the necessity and universality of mathematical constructs.
- Justificationism: It includes arguments as a primary object type. The arguments can be descriptive, explanatory, and justificatory, and use different types of reasoning, based on both reason and experience.

Semiotic dimension

- Realism: In realistic theories of meaning (KUTSCHERA, 1975), linguistic expressions have a relation of attribution to certain entities (objects, attributes, facts). Words and signs are made meaningful by the fact that an object, a concept, or a proposition as meaning are assigned to them. In this way, there are entities, not necessarily concrete, but always objectively given prior to the words, which are their meanings. The OSA postulates a type of semiotic function that is referential, designating certain entities under conventions. This is how the representational value of languages is accounted for.
- Pragmatism: In pragmatic (operational) theories, the meaning depends on the context in which words are used. Signs become meaningful by the fact of playing a certain function in a linguistic game, the fact being used in this game in a certain way, and for a certain purpose. The meanings of mathematical objects as systems of operative and discursive practices imply the acceptance of the postulates of pragmatic theories and the recognition of the instrumental value of languages.

The OSA assigns an essential role to the creation and manipulation of sign systems as means of representation of different types of objects and as instruments of mathematical activity. Hence, semiotic-cognitive theories consider representationist and instrumentalist postulates as compatible and complementary.

5. Final reflections

The OSA provides a transdisciplinary vision of mathematical activity by considering, in an articulated manner, different points of view from the disciplines interested in mathematical knowledge, and in its teaching and learning. The following disciplines should be considered: Epistemology: Mathematics as a particular mode of human activity and its product as a special type of knowledge; Ontology: Mathematics as a finished product, i.e., a system of objects and theories; Psychology: Mathematics as a particular type of mental (or cerebral) activity; Sociology: Mathematics as a type of social activity and its product as a special type of cultural artefact; History: Mathematics as a historical process of discovery, invention, and diffusion in a particular society; Instrumental point of view: Mathematics as a tool for science, technology, and humanities.

These different ways of looking at mathematics are mutually compatible, even complementary. It would be wrong to adopt one of them to the exclusion of all others since mathematics is, at the same time, all that these different views provide.

Several authors have developed constructs and theories to respond to the epistemological, ontological, and semiotic-cognitive problems described in this paper as specific to educational mathematics. The study of the concordances and complementarities with other theories of the model proposed by the OSA has been addressed in previous research works[4]. In particular, the comparison with the anthropological theory of didactics (CHEVALLARD, 1992), APOS theory (DUBINSKY; MCDONALD, 2001), objectification theory (RADFORD, 2014), semiotic representation registers (DUVAL, 1995), among others, has been addressed. These studies of articulation of theoretical frameworks will need to be extended in future research, particularly concordances with Sfard's (2008) framework of "commognition".

The foundations of educational mathematics described in section 4 are being used to develop tools to address issues related to the design, implementation, and evaluation of mathematics instructional processes. The constructs, institutional and personal meanings, understood in pragmatic terms, and the proposed types of meanings, provide criteria for curriculum and lesson design (GODINO et al., 2014). In order to address issues related to the

[4] Available in the "*Articulation of theoretical frameworks*" section in the web repository http://enfoqueontosemiotico.ugr.es

analysis of implementing instructional processes, the didactic configuration tool has been developed (GODINO; CONTRERAS; FONT, 2006). Likewise, the theory of didactic suitability (BREDA; FONT; PINO-FAN, 2018; GODINO, 2013) addresses questions regarding the assessment of instructional processes and teacher education. All these tools are supported by the onto-semiotic modelling of mathematical knowledge.

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